inant, it is generally not very effective. A slightly more sophisticated splitting sets $M = (L + D)$ and $K = -U$. Because M is lower triangular, each iteration of the resulting *Gauss–Seidel* preconditioner is easily found by back–substitution. Finally, the *successive overrelaxation* (SOR) method attempts to accelerate the Gauss–Seidel iteration by adjusting the step size. For more details on these stationary methods, see [7, 25, 37, 85]. Note, however, that none of the classical stationary iterations are competitive with the conjugate gradient method presented in the following section.

2.4.2 The Conjugate Gradient Iteration

Krylov subspace methods, of which the *conjugate gradient* (CG) iteration is a special case, are a class of nonstationary iterative methods commonly used to solve sparse linear systems. The Krylov subspace of dimension k generated by a matrix A and vector r is defined as

$$\mathcal{K}_k(A, r) \triangleq \text{span}\left(r, Ar, A^2r, \ldots, A^{k-1}r\right) \tag{2.52}$$

Krylov subspace methods compute a sequence of iterates x^n which optimize some objective function over Krylov subspaces of successively increasing dimension. The conjugate gradient method examined in this section is by far the most widely used Krylov subspace method for positive definite systems. For more details on CG and other Krylov subspace methods, see [7, 25, 37, 46].

The CG iteration is based on the Krylov subspaces generated by A and an initial residual $r^0 = b - Ax^0$, where x^0 is some initial guess for the solution of equation (2.46). At the n^{th} iteration, CG selects the vector x^n as

$$\begin{aligned} x^n &= \underset{\bar{x} \in x^0 \oplus \mathcal{K}_n(A, r^0)}{\arg\min} \left(A\bar{x} - b\right)^T A^{-1} \left(A\bar{x} - b\right) \\ &= \underset{\bar{x} \in x^0 \oplus \mathcal{K}_n(A, r^0)}{\arg\min} ||A\bar{x} - b||_{A^{-1}} \end{aligned} \tag{2.53}$$

CG is particularly effective because the minimization in equation (2.53) can be performed very efficiently, requiring only a few matrix–vector products involving the