The dynamic behavior of equation (2.48) depends heavily on A itself. In particular, convergence will be fastest when $\{\lambda_i(A)\}$ are concentrated around 1, while the iteration will diverge if $\rho(A) > 2$. To correct this problem, we can transform the linear system (2.46) into an alternate system which has the same solution, but more favorable spectral properties. In particular, for any invertible matrix M, we may write

$$M^{-1}Ax = M^{-1}b \tag{2.49}$$

The matrix M is known as a *preconditioner* [7, 25], and naturally leads to the following preconditioned Richardson iteration:

$$x^n = (I - M^{-1}A)x^{n-1} + M^{-1}b \tag{2.50}$$

If M is chosen so that the eigenvalues of $M^{-1}A$ are closer to 1 than those of the original system A, the preconditioned iteration (2.50) may converge after a smaller number of iterations. However, each iteration is more costly, because it is necessary to multiply Ax^{n-1} by M^{-1}, or equivalently to solve a linear system of the form $M\bar{x} = \bar{b}$. Thus, in addition to well approximating A, an effective preconditioner must ensure that each evaluation of equation (2.50) is not too difficult.

Preconditioners are frequently generated using a *matrix splitting* [25, 37, 46, 85] $A = M - K$ of the linear system A, where K is chosen so that $M = A+K$ is invertible. For any such splitting, the corresponding preconditioned Richardson iteration is

$$\begin{aligned} x^n &= \left(I - (A+K)^{-1}A\right)x^{n-1} + (A+K)^{-1}b \\ &= (A+K)^{-1}\left(Kx^{n-1} + b\right) \end{aligned} \tag{2.51}$$

Many classic iterative algorithms are generated by appropriately chosen splittings. Let $A = L + D + U$ where L, D, and U are lower triangular, diagonal, and upper triangular, respectively. The *Gauss–Jacobi* method chooses $M = D$ and $K = -(L+U)$. The resulting preconditioner is particularly easy to apply, simply requiring Ax^{n-1} to be rescaled by D^{-1} at each iteration. However, unless A is strongly diagonally dom-