Loop Series: A Key Identity

$$Z(q) = \sum_{x \in \{0,1\}^n} \prod_{s \in V} q_s(x_s) \prod_{(s,t) \in E} \frac{q_{st}(x_s, x_t)}{q_s(x_s) q_t(x_t)}$$

- For binary variables, reparameterized pairwise compatibilities can be expressed as follows:

$$\frac{q_{st}(x_s, x_t)}{q_s(x_s) q_t(x_t)} = 1 + \beta_{st}(x_s - \tau_s)(x_t - \tau_t)$$

$$\beta_{st} := \frac{\tau_{st} - \tau_s \tau_t}{\tau_s(1 - \tau_s)\tau_t(1 - \tau_t)} = \frac{\mathrm{Cov}_{q_{st}}(X_s, X_t)}{\mathrm{Var}_{q_s}(X_s)\,\mathrm{Var}_{q_t}(X_t)}$$

- Straightforward (but tedious) to verify for $(x_s, x_t) \in \{0,1\}^2$
- For *attractive* compatibilities, note that $\beta_{st} \geq 0$