dependent CRP resulting from this f and any collection of sequential distances is marginally invariant. Then the probability that customers 1 and 3 share a table must be the same whether customer 2 is absent or present.

If customer 2 is absent,

$$\mathbb{P}\{1 \text{ and } 3 \text{ sit at same table} \mid 2 \text{ absent}\} = \frac{f(d_{31})}{f(d_{31}) + \alpha}. \tag{11}$$

If customer 2 is present, customers 1 and 3 may sit at the same table in two different ways: 3 sits with 1 directly ($c_3 = 1$); or 3 sits with 2, and 2 sits with 1 ($c_3 = 2$ and $c_2 = 1$). Thus,

$$\begin{aligned}&\mathbb{P}\{1 \text{ and } 3 \text{ sit at same table} \mid 2 \text{ present}\} \\ &\qquad = \frac{f(d_{31})}{f(d_{31}) + f(d_{32}) + \alpha} + \left(\frac{f(d_{32})}{f(d_{31}) + f(d_{32}) + \alpha}\right)\left(\frac{f(d_{21})}{f(d_{21}) + \alpha}\right).\end{aligned} \tag{12}$$

For the distance dependent CRP to be marginally invariant, Eq. (11) and Eq. (12) must be identical. Writing Eq. (11) on the left side and Eq. (12) on the right, we have

$$\frac{f(d_{31})}{f(d_{31}) + \alpha} = \frac{f(d_{31})}{f(d_{31}) + f(d_{32}) + \alpha} + \left(\frac{f(d_{32})}{f(d_{31}) + f(d_{32}) + \alpha}\right)\left(\frac{f(d_{21})}{f(d_{21}) + \alpha}\right). \tag{13}$$

We now consider two different possibilities for the distances d_{32} and d_{21}, always keeping $d_{31} = d_{21} + d_{32}$.

First, suppose $d_{21} = 0$ and $d_{32} = d_{31} = d$ for some $d \geq 0$. By multiplying Eq. (13) through by $(2f(d) + \alpha)(f(0) + \alpha)(f(d) + \alpha)$ and rearranging terms, we obtain

$$0 = \alpha f(d)\left(f(0) - f(d)\right).$$

Thus, either $f(d) = 0$ or $f(d) = f(0)$. Since this is true for each $d \geq 0$ and f is nonincreasing, $f = a1[d \in A]$ with $a \geq 0$ and either $A = \emptyset$, $A = \mathbb{R}$, $A = [0, b]$, or $A = [0, b)$ with $b \in [0, \infty)$. Because $A = \emptyset$ is among the choices, we may assume $a > 0$ without loss of generality. We now show that if $A = [0, b]$ or $A = [0, b)$, then we must have $b = 0$ and A is of the form claimed by the proposition.

Suppose for contradiction that $A = [0, b]$ or $A = [0, b)$ with $b > 0$. Consider distances given by $d_{32} = d_{21} = d = b - \epsilon$ with $\epsilon \in (0, b/2)$. By multiplying Eq. (12) through by

$$\left(f(2d) + f(d) + \alpha\right)\left(f(d) + \alpha\right)\left(f(2d) + \alpha\right)$$

and rearranging terms, we obtain

$$0 = \alpha f(d)\left(f(d) - f(2d)\right).$$

Since $f(d) = a > 0$, we must have $f(2d) = f(d) > 0$. But, $2d = 2(b - \epsilon) > b$ implies together with $f(2d) = a1[2d \in A]$ that $f(2d) = 0$, which is a contradiction. ■

These two propositions are combined in the following corollary, which states that the class of decay functions considered in Propositions 1 and 2 is both necessary and sufficient for marginal invariance.