CRPs, where the kth distance dependent CRP governs linkages between customers in B_k using

$$p(c_i = j) \propto \begin{cases} \alpha & \text{if } i = j, \\ a & \text{if } j < i, \\ 0 & \text{if } j > i, \end{cases}$$

for $i, j \in B_k$.

Finally, dividing the unnormalized probabilities by a, we rewrite the linkage probabilities for the kth distance dependent CRP as

$$p(c_i = j) \propto \begin{cases} \alpha/a & \text{if } i = j, \\ 1 & \text{if } j < i, \\ 0 & \text{if } j > i, \end{cases}$$

for $i, j \in B_k$. This is identical to the distribution of the traditional CRP with concentration parameter α/a.

This shows that the distance dependent CRP with decay function $f(d) = a1[d \in A]$ induces the same probability distribution on clusters as the one produced by a collection of K independent traditional CRPs, each with concentration parameter α/a, where the kth traditional CRP governs the clusters of customers within B_k.

The marginal invariance of this distribution follows from the marginal invariance of each traditional CRP, and their independence from one another. ■

The probability distribution described in this proposition separates customers into groups $B_1, \ldots, B_K$ based on whether inter-customer distances fall within the set A, and then governs clustering within each group independently using a traditional CRP. Clustering across groups does not occur.

We consider what this means for specific choices of A. If $A = \{0\}$, then each group contains those customers whose distance from one another is 0. This group is well-defined because of the assumption that $d_{ij} = 0$ and $d_{jk} = 0$ implies $d_{ik} = 0$. If $A = \mathbb{R}$, then each group contains those customers whose distance from one another is finite. Similarly to the $A = \{0\}$ case, this group is well-defined because of the assumption that $d_{ij} < \infty$ and $d_{jk} < \infty$ implies $d_{ik} < \infty$. If $A = \emptyset$, then each group contains only a single customer. In this case, each customer will be in his own cluster.

Since the resulting construction is marginally invariant, Proposition 1 provides a sufficient condition for marginal invariance. The following proposition shows that this condition is necessary as well.

Proposition 2 *If the distance dependent CRP for a given decay function f is marginally invariant over all sets of sequential distances then f is of the form $f(d) = a1[d \in A]$ for some $a > 0$ and A equal to either $\emptyset$, $\{0\}$, or $\mathbb{R}$.*

Proof Consider a setting with 3 customers, in which customer 2 may either be absent, or present with his seating assignment marginalized out. Fix a non-increasing decay function f with $f(\infty) = 0$ and suppose that the distances are sequential, so $d_{13} = d_{23} = d_{12} = \infty$. Suppose that the distance