we write

$$\rho_i \equiv \frac{a_i}{1+a_i}, \qquad \pi_i \equiv \frac{a_{i+1}p_{i+1}}{(1+a_i)\,p_i}$$

Hence, ρ_i is proportional to the receivables to sales ratio, while π_i is proportional to the payables to sales ratio. Hence, ε is the elasticity of ρ_i with respect to π_i. Let

$$\varepsilon\,(i,i+1) \equiv \frac{\left(\rho_{i+1}-\rho_i\right)/\rho_i}{\left(\pi_{i+1}-\pi_i\right)/\pi_i} \tag{22}$$

$\varepsilon\,(i,i+1)$ is the elasticity of the receivables to sales ratio with respect to the payables to sales ratio going upstream from firm i to firm $i+1$. Using our solution for the optimal contract,

$$\begin{aligned} \varepsilon\,(i,i+1) &= \frac{\frac{a_{i+1}}{1+a_{i+1}}\frac{1+a_i}{a_i}-1}{\frac{a_{i+2}p_{i+2}}{a_{i+1}p_{i+1}}\frac{(1+a_i)p_i}{(1+a_{i+1})p_{i+1}}-1} \\ &= \frac{\frac{a_{i+1}}{a_i}-1}{(1+a_i)\,p_i\frac{a_{i+2}p_{i+2}}{a_{i+1}p_{i+1}^2}-1-a_{i+1}} \end{aligned} \tag{23}$$

We are interested in how $\varepsilon\,(i,i+1)$ changes as w_i becomes large. We know from our solutions for a_k and p_k that all quantities with subscripts $i+1$ and higher are unaffected by changes in w_i. As w_i becomes large, the numerator of (23) is bounded away from zero. However, the denominator of (23) increases without bound. Hence $\varepsilon\,(i,i+1) \to 0$ as $w_i \to 0$.

In other words, as the size of firm i increases for any fixed profile of firm sizes upstream from i, the elasticity $\varepsilon\,(i,i+1)$ becomes small. By selecting a profile for firm sizes $\{w_i\}$ that increases sufficiently quickly going downstream in the production chain, we can make the elasticities $\{\varepsilon\,(i,i+1)\}$ as low as we please.

In particular, if we confine our attention to size profiles $\{w_i\}$ that give rise to constant elasticities along the production chain so that $\varepsilon\,(i,i+1)=\varepsilon$, for all i, a low cross-section elasticiy ε is associated with a high degree of vertical integration. Our reasoning suggests the following hypothesis.