## 3.1 Optimal Pattern of Delays

We can now solve for the pattern of delays in the optimal contract by solving for the ratio of receivables to sales along the production chain. The ratio of receivables to sales is proportional to $\frac{a_i p_i}{(1+a_i)p_i}$, which is just $\frac{a_i}{1+a_i}$. We have the following solution.

**Lemma 1** *In the optimal contract,*

$$\frac{a_i}{1+a_i} = \frac{\sum_{k=i}^{N} \beta_k w_k}{\sum_{k=i}^{N} (1+b_k)\, w_k} \tag{17}$$

We prove this by induction, backward from $N$. The lemma holds for $i = N$. For the inductive step, note that from (15) and the fact that the IC constraint (8) binds with equality, we can write

$$\frac{a_i}{1+a_i} = \frac{a_{i+1}}{1+a_{i+1}} \frac{\sum_{k=i+1}^{N} (1+b_k)\, w_k}{\sum_{k=i}^{N} (1+b_k)\, w_k} + \frac{\beta_i w_i}{\sum_{k=i}^{N} (1+b_k)\, w_k} \tag{18}$$

By the induction hypothesis, $a_{i+1}/\left(1+a_{i+1}\right) = \sum_{k=i+1}^{N} \beta_k w_k / \sum_{k=i+1}^{N} (1+b_k)\, w_k$. Substituting into (18) and simplifying gives (17). This proves lemma 1.

For the case where $\beta_k \approx b_k$ equation (17) gives the approximation

$$a_i \approx \frac{\sum_{k=i}^{N} b_k w_k}{\sum_{k=i}^{N} w_k} \tag{19}$$

In other words, the amortization coefficient $a_i$ is a weighted average of all the moral hazard parameters $\{b_k\}$ upstream from $i$, where the weights are given by the relative size of each firm in terms of wage costs. Since $b_k \leq b_{k+1}$, we have

$$a_i \leq a_{i+1} \leq \cdots \leq a_N \tag{20}$$

In other words, the delay is longer for firms that are further upstream, reflecting the fact that upstream firms are more remote from the final product.