chapter 2

Partial Differential Equations for Fluids

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Denote by Ω the region of space (R^3) occupied by the fluid. Denote by $(0, T^1)$ the time interval of interest. A Newtonian fluid is characterized by

- a density field $\rho(x,t)$,
- a velocity vector field $u(x,t)$,
- a pressure field $p(x,t)$,
- a temperature field $T(x,t)$,

for all $(x,t) \in \Omega \times (0, T^1)$.

2.1 Conservation of Mass

The variation of mass of fluid in $\mathcal{O}$ has to be equal to the mass flux across the boundaries of $\mathcal{O}$. So if n denotes the exterior normal to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$,

$$\partial_t \int_{\mathcal{O}} \rho = -\int_{\partial\mathcal{O}} \rho u \cdot n,$$

By using Stokes' formula

$$\int_{\mathcal{O}} \nabla.(\rho u) = \int_{\partial\mathcal{O}} \rho u \cdot n$$

and the fact that $\mathcal{O}$ is arbitrary, the *continuity equation* is obtained:

$$\partial_t \rho + \nabla.(\rho u) = 0. \quad (1)$$

2.2 Conservation of momentum

The forces on $\mathcal{O}$ are the external forces f (gravity for instance) and the force that the fluid outside $\mathcal{O}$ exercises, $\sigma n - pn$ per volume element, by definition of the stress tensor σ. Hence Newton's law, written for a volume element $\mathcal{O}$ of fluid gives

$$\int_{\mathcal{O}} \rho \frac{du}{dt} = \int_{\partial\mathcal{O}} (\sigma\mathbf{n} - pn)$$

Now

$$\begin{aligned} \frac{du}{dt}(x,t) &= \lim_{\delta t \to 0} \frac{1}{\delta t}[u(x + u(x,t)\delta t, t + \delta t) - u(x,t)] \\ &= \partial_t u + \sum_j u_j \partial_j u \equiv \partial_t u + u\nabla u \\ & \rho(\partial_t u + u\nabla u) + \nabla p - \nabla.\sigma = f. \end{aligned}$$

By the continuity equation, this equation is equivalent to the *momentum equation*:

$$\partial_t(\rho u) + \nabla.(\rho u \otimes u) + \nabla.(p\mathbf{I} - \sigma) = f. \qquad (2)$$

2.2.1 Newtonian flow

To proceed further an hypothesis is needed to relate the stress tensor σ to u. For *Newtonian flows* is a linear relation is assumed:

$$\sigma = \mu(\nabla u + \nabla u^T) + (\xi - \frac{2\mu}{3})\mathbf{I}\nabla.u \qquad (3)$$

The scalars μ and ξ are called the *first and second viscosities* of the fluid. For air and water the second viscosity ζ is very small, so $\xi = 0$ and the equation of momentum becomes

$$\partial_t(\rho u) + \nabla.(\rho u \otimes u) + \nabla p - \nabla.[\mu(\nabla u + \nabla u^T) - \frac{2\mu}{3}\mathbf{I}\nabla.u] = f. \qquad (4)$$

2.3 Conservation of energy and the law of state

Conservation of energy is obtained by writing that the variation of the total energy in a volume element balances heat variation and the work of forces $\mathcal{O}$

The energy $E(x,t)$ per unit mass in a volume element $\mathcal{O}$ is the sum of the internal energy e and the kinetic energy $u^2/2$.

The work done by the forces is the integral over $\mathcal{O}$ of $u \cdot (f + \sigma - p\mathbf{I})n$.

By definition of the temperature T, if there is no heat source (combustion...) the amount of heat received (lost) is proportional to the flux of the temperature gradient, i.e. the integral on $\partial\mathcal{O}$ of $\kappa\nabla T.n$. The scalar κ is called the *thermal conductivity.* So the following equation is obtained

$$\frac{d}{dt}\int_{\mathcal{O}(t)}\rho E = \int_{\mathcal{O}}\{\partial_t \rho E + \nabla.[u\rho E]\} = \int_{\mathcal{O}} u\cdot f + \int_{\partial\mathcal{O}}[u(\sigma - p\mathbf{I}) - \kappa\nabla T]n$$

With the continuity equation and Stokes' formula it is transformed into

$$\partial_t[\rho E] + \nabla.(u[\rho E + p]) = \nabla.(u\sigma + \kappa\nabla T) + f\cdot u$$

To close the system a definition for e is needed. For an *ideal fluid* , such as air and water in non extreme situations, C_v and C_p being physical constants, we have

$$e = C_v T, \quad E = C_v T + \frac{u^2}{2}, \tag{5}$$

and the *equation of state*

$$\frac{p}{\rho} = RT, \tag{6}$$

where R is an ideal gas constant. With $\gamma = C_p/C_v = R/C_v + 1$, the above can be written as,

$$e = \frac{p}{\rho(\gamma - 1)}. \tag{7}$$

With the definition of σ, the equation for E becomes what is usually referred as *the energy equation*:

$$\partial_t[\rho\frac{u^2}{2} + \frac{p}{\gamma-1}] + \nabla.\{u[\rho\frac{u^2}{2} + \frac{\gamma}{\gamma-1}p]\} \tag{8}$$

$$= \nabla.\{\kappa\nabla T + [\mu(\nabla u + \nabla u^T) - \frac{2}{3}\mu\mathbf{I}\nabla.u]u\} + f\cdot u \tag{9}$$

By introducing the *entropy* :

$$s \equiv \frac{R}{\gamma - 1}\log\frac{p}{\rho^\gamma}, \tag{10}$$

another form of the energy equation is the entropy equation:

$$\rho T(\partial_t s + u\nabla s) = \frac{\mu}{2}|\nabla u + \nabla u^T|^2 - \frac{2}{3}\mu|\nabla.u|^2 + \kappa\Delta T.$$

3 Inviscid Flows

In many instances viscosity plays a limited effect. If it is neglected, together with the temperature diffusion, ($\kappa = 0, \eta = \xi = 0$) the equations for the fluid become the *Euler equations*:

$$\partial_t \rho + \nabla.(\rho u) = 0 \tag{11}$$

$$\partial_t[\rho \frac{u^2}{2} + \frac{p}{\gamma - 1}] + \nabla.\{u[\rho \frac{u^2}{2} + \frac{\gamma}{\gamma - 1} p]\} = f \cdot u \tag{12}$$

Notice also the in the absence of shocks the equation for the entropy (10) becomes

$$\frac{\partial s}{\partial t} + u \nabla s = 0$$

hence s is constant on the lines tangent at each point to u (stream lines). In fact a stream line is a solution of the equation :

$$x'(\tau) = u(x(\tau), \tau)$$

and so

$$\frac{d}{dt} s(x(t), t) = \frac{\partial s}{\partial x_i} \frac{\partial x_i}{\partial t} + \frac{\partial s}{\partial t} = \partial_t s + u \nabla s = 0.$$

If s is constant and equal to s^0 constant at time 0 and if s is also equal to s^0 on the part of Γ where $u \cdot n < 0$, then there is an analytical solution $s = s^0$.

Finally there remains a system of two equations with two unknowns, for *isentropic flows*

$$\partial_t \rho + \nabla.(\rho u) = 0 \tag{13}$$

$$\rho(\partial_t u + u \nabla u) + \nabla p = f \tag{14}$$

$$\text{where } p = C \rho^\gamma \quad (C = e^{s^0 \frac{\gamma - 1}{R}}). \tag{15}$$

4 Incompressible Flows

When the variations of ρ are small (water for example or air at low velocity) we can neglect its derivatives. Then the general equations become the incompressible *Navier Stokes equations*

$$\nabla.u = 0, \tag{16}$$

$$\partial_t u + u \nabla u + \nabla p - \nu \Delta u = f/\rho, \tag{17}$$

with $\nu = \mu/\rho$ the *kinematic viscosity* and $p \to p/\rho$ the reduced pressure. If buoyancy effects are present in f, we need an equation for the temperature.

An equation for the temperature T can be obtained from the energy equation

$$\partial_t T + u \nabla T - \frac{\kappa}{\rho C_v} \Delta T = \frac{\nu}{2 C_v} |\nabla u + \nabla u^T|^2. \tag{18}$$

5 Potential Flows

For suitable boundary conditions the solution of the Navier-Stokes equations can be irrotational

$$\nabla \times u = 0$$

By the theorem of De Rham there exists then a *potential* function such that

$$u = \nabla\varphi$$

Using the identities :

$$\Delta u = -\nabla \times \nabla \times u + \nabla(\nabla . u) \tag{19}$$

$$u\nabla u = -u \times (\nabla \times u) + \nabla(\frac{u^2}{2}) \tag{20}$$

we see,that when u is a solution of the incompressible Navier-Stokes equations (17) and $f = 0$, φ is a solution of the *Laplace equation* :

$$\Delta\varphi = 0, \tag{21}$$

and the *Bernoulli* equation, derived from (17), gives the pressure

$$p = k - \frac{1}{2}|\nabla\varphi|^2. \tag{22}$$

This type of flow is the simplest of all.
In the same way, with isentropic inviscid flow (13)

$$\partial_t\rho + \nabla\varphi\nabla\rho + \rho^0\Delta\varphi = 0$$
$$\nabla(\varphi_{,t} + \frac{1}{2}|\nabla\varphi|^2 + \gamma C\rho^{0\gamma-1}\rho) = 0$$

If we neglect the convection term $\nabla\varphi\nabla\rho$ this system simplifies to a nonlinear *wave equation* :

$$\partial_{tt}\varphi - c\Delta\varphi + \frac{1}{2}\partial_t|\nabla\varphi|^2 = d(t) \tag{23}$$

where $c = \gamma C\rho^{0\gamma}$ is related to the velocity of the sound in the fluid.

Finally, we show that there are stationary potential solutions of Euler equations (12) with $f = 0$. Using (19) the equations can be rewritten as

$$-\rho u \times \nabla \times u + \rho\nabla\frac{u^2}{2} + \nabla p = 0.$$

Taking the scalar product with u, we obtain

$$u \cdot [\rho \nabla \frac{u^2}{2} + \nabla p] = 0$$

Also, the pressure being given by (13)

$$u \cdot (\rho \nabla \frac{u^2}{2} + \rho^{\gamma-1} C \gamma \nabla \rho) = 0$$

Or equivalently

$$u\rho . \nabla (\frac{u^2}{2} + C \frac{\gamma}{\gamma - 1} \rho^{\gamma-1}) = 0$$

So the quantity between the parenthesis is constant along the stream lines, that is we have

$$\rho = \rho^0 (k - \frac{u^2}{2})^{\frac{1}{\gamma-1}}$$

Indeed the solution of the PDE $u \nabla \xi = 0$, in the absence of shocks, is ξ constant on the stream lines. If it is constant upstream (on the inflow part of the boundary , $u \cdot n < 0$), and if there are no closed stream lines then ξ is constant everywhere. Thus if ρ^0 and k are constant upstream then $\nabla \times u$ is parallel to u and, at least in 2 dimensions , this implies that u derives from a potential. The *transsonic potential flow equation.* is then obtained

$$\nabla . [(k - |\nabla \varphi|^2)^{\frac{1}{\gamma-1}} \nabla \varphi] = 0 \tag{24}$$

The time dependent version of this equation is obtained from

$$u = \nabla \varphi, \quad \rho = \rho^0 (k - (\frac{u^2}{2} + \partial_t \varphi)^{\frac{1}{\gamma-1}}, \quad \partial_t \rho + \nabla \cdot (\rho u) = 0$$

6 Turbulence modeling

In this section we consider the Navier-Stokes equations for incompressible flows (17)

6.1 The Reynolds Number

let us rewrite (17) in non dimensional form.

Let U be a characteristic velocity of the flow under study (for example one of the non homogeneous boundary conditions). Let L be a characteristic length (for example the diameter of Ω) and T_1 a characteristic time (which is a priori equal to L/U). Let us put

$$u' = \frac{u}{U}; \quad x' = \frac{x}{L}; \quad t' = \frac{t}{T_1}.$$

Then (17) can be rewritten as

$$\nabla_{x'} \cdot u' = 0, \quad \frac{L}{T_1 U}\partial_{t'} u' + u'\nabla_{x'} u' + \frac{1}{U^2}\nabla_{x'} p - \frac{\nu}{LU}\Delta_{x'} u' = f\frac{L}{U^2}.$$

So, if we put $T_1 = L/U$, $p' = p/U^2$, $\nu' = \nu/LU$, then the equations are the same but with "prime" variables. The inverse of ν' is called the *Reynolds number.*

$$Re = \frac{UL}{\nu}.$$

6.2 Reynolds Equations

Consider (17) with random initial data, $u^0 = \overline{u}^0 + u'^0$, where $\overline{u}$ stands for the expected value.

Taking the expected value of the Navier-Stokes equations leads to

$$\nabla \cdot \overline{u} = 0,$$
$$\partial_t \overline{u} + \nabla \cdot \overline{(\overline{u} + u') \otimes (\overline{u} + u')} + \nabla \overline{p} - \nu \Delta \overline{u} = \overline{f},$$

which is also

$$\nabla \cdot \overline{u} = 0, \quad R = -\overline{u' \otimes u'}, \tag{25}$$
$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \nabla \overline{p} - \nu \Delta \overline{u} = \overline{f} + \nabla \cdot R. \tag{26}$$

6.3 The $k - \epsilon$ model

Reynolds hypothesis is that the turbulence in the flow is a local function of $\nabla \overline{u} + \nabla \overline{u}^T$:

$$R(x,t) = R(\nabla \overline{u}(x,t) + \nabla \overline{u}^T(x,t))$$

If the turbulence is locally isotropic at scales smaller than those described by the model (26) and if the Reynolds hypothesis holds then it is reasonable to express R on the basis formed with the powers of $\nabla \overline{u} + \nabla \overline{u}^T$ and to relate the coefficients to the two turbulent quantities used by Kolmogorov to characterize homogeneous turbulence: the kinetic energy of the small scales k and their rate of viscous energy dissipation ε

$$k = \frac{1}{2}\overline{|u'|^2}, \qquad \varepsilon = \frac{\nu}{2}\overline{|\nabla u' + \nabla u'^T|^2}$$

For two dimensional mean flows (for some $\alpha(x,t)$)

$$R = \nu_T(\nabla\overline{u} + \nabla\overline{u}^T) + \alpha I, \quad \nu_T = c_\mu \frac{k^2}{\epsilon} \tag{27}$$

and k and ε are modeled by

$$\partial_t k + \overline{u}\nabla k - \frac{c_\mu}{2}\frac{k^2}{\epsilon}|\nabla\overline{u} + \nabla\overline{u}^T|^2 - \nabla.(c_\mu \frac{k^2}{\epsilon}\nabla k) + \epsilon = 0, \tag{28}$$

$$\partial_t \epsilon + \overline{u}\nabla\epsilon - \frac{c_1}{2}k|\nabla\overline{u} + \nabla\overline{u}^T|^2 - \nabla.(c_\epsilon \frac{k^2}{\epsilon}\nabla\epsilon) + c_2\frac{\epsilon^2}{k} = 0, \tag{29}$$

with $c_\mu = 0.09$, $c_1 = 0.126$, $c_2 = 1.92$, $c_\epsilon = 0.07$.

The model is derived heuristically from the Navier-Stokes equations with the following hypotheses:

- frame invariance and 2D mean flow, ν_T a polynomial function of k, ϵ.
- u'^2 and $|\nabla \times u'|^2$ are passive scalars when convected by $\overline{u} + u'$.
- Ergodicity allows statistical averages to be replaced by space averages.
- Local isotropy of the turbulence at the level of small scales.
- A Reynolds hypothesis for $\overline{\nabla \times u' \otimes \nabla \times u'}$.
- A closure hypothesis : $\overline{|\nabla \times \nabla \times u'|^2} = c_2\ \epsilon^2/k$.

The constants $c_\mu, c_\epsilon, c_1, c_2$ are chosen so that the model reproduces

- The decay in time of homogeneous turbulence
- The measurements in shear layers in local equilibrium
- The log wall law in boundary layers.

The model is *not valid near solid walls* because the turbulence is not isotropic so the near wall boundary layers are removed from the computational domain. An adjustable artificial boundary is placed parallel to the walls Γ at a distance $\delta(x,t) \in [10, 100]\nu/u_\tau$.
A possible set of boundary conditions is then

$$\overline{u}, k, \epsilon \quad \text{given everywhere at t = 0 ,}$$

$$\overline{u}, k, \epsilon \quad \text{given on the inflow boundaries at all t,}$$

$$\nu_T\partial_n\overline{u}, \nu_T\partial_n k, \nu_T\partial_n\epsilon \ \ \text{given on outflow boundaries (usually zero) at all t}$$

$$u \cdot n = 0, \quad \frac{\overline{u} \cdot s}{\sqrt{\nu|\partial_n \overline{u}|}} - \frac{1}{\chi}\log(\delta\sqrt{\frac{1}{\nu}|\partial_n \overline{u}|}) + \beta = 0,$$

$$k|_{\Gamma+\delta} = |\nu\partial_n(\overline{u} \cdot s)|c_\mu^{-\frac{1}{2}}, \quad \epsilon|_{\Gamma+\delta} = \frac{1}{\chi\delta}|\nu\partial_n(\overline{u} \cdot s)|^{\frac{3}{2}}$$

where $\chi = 0.41, \beta = 5.5$ for smooth walls, n, s are the normal and tangent to the wall and δ is a function such that at each point of $\Gamma + \delta$,

$$10\sqrt{\nu/|\partial_n(\overline{u} \cdot s)|} \le \delta \le 100\sqrt{\nu/|\partial_n(\overline{u} \cdot s)|}.$$

These wall functions are classical and are only valid for regions where the flow is attached and the turbulence fully developed in the boundary layer. Bellow, we show how to derive generalized wall functions valid up to the wall and also valid for separated and unsteady flows.

7 Equations for Compressible Flows in Conservation Form

For compressible flows, let us consider the conservation form of the Navier-Stokes equations with the $k - \varepsilon$ model. As it has been, the model is derived by splitting the variables into mean and fluctuating parts and use Reynolds averages for density and pressure and Favre averages for other variables. The nondimensionalized Reynolds averaged equations are closed by an appropriate modeling [1], we have:

$$\partial_t\rho + \nabla \cdot (\rho u) = 0$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla(p + \frac{2}{3}\rho k) = \nabla \cdot ((\mu + \mu_t)S)$$

$$\partial_t(\rho E) + \nabla \cdot ((\rho E + p + \frac{5}{3}\rho k)u) = \nabla \cdot ((\mu + \mu_t)Su) + \nabla((\chi + \chi_t)\nabla T)$$

with

$$\chi = \frac{\gamma\mu}{Pr}, \quad \chi_t = \frac{\gamma\mu_t}{Pr_t},$$

$$\gamma = 1.4, \quad Pr = 0.72 \quad \text{and} \quad Pr_t = 0.9\,,$$

where μ and μ_t are the inverse of the laminar and turbulent Reynolds numbers. In what follows, we call them viscosity. The laminar viscosity μ is given by Sutherland's law :

$$\mu = \mu_\infty(\frac{T}{T_\infty})^{1.5}(\frac{T_\infty + 110.4}{T + 110.4}), \tag{30}$$

where f_∞ denotes a reference quantity for f or its value at infinity if the flow is uniform there.

$$\mathbf{S} = (\mu + \mu_t)(\nabla u + \nabla u^T - \frac{2}{3}\nabla.uI)$$

is the Newtonian strain tensor.

We consider the state equation for perfect gas:

$$p = (\gamma - 1)\rho T.$$

Experience shows that almost everywhere $\rho k << p$, we therefore drop the turbulent energy contributions in terms with first order derivative (the hyperbolic part). This improves also the numerical stability, reducing the coupling between the equations [1].

The $k - \varepsilon$ model [2] we use is classical; it is an extension to compressible flows of the incompressible version [3], (see Chapter 2):

$$\partial_t \rho k + \nabla.(\rho u k) - \nabla((\mu + \mu_t)\nabla k) = S_k, \tag{31}$$

and

$$\partial_t \rho \varepsilon + \nabla.(\rho u \varepsilon) - \nabla((\mu + c_\varepsilon \mu_t)\nabla \varepsilon) = S_\varepsilon. \tag{32}$$

The right hand sides of (31)-(32) contain the production and the destruction terms for ρk and $\rho\varepsilon$:

$$S_k = \mu_t P - \frac{2}{3}\rho k \nabla.u - \rho\varepsilon, \tag{33}$$

$$S_\varepsilon = c_1 \rho k P - \frac{2c_1}{3c_\mu}\rho\varepsilon\nabla.u - c_2\rho\frac{\varepsilon^2}{k}. \tag{34}$$

The eddy viscosity is given by:

$$\mu_t = c_\mu \rho \frac{k^2}{\varepsilon}. \tag{35}$$

The constants $c_\mu, c_1, c_2, c_\varepsilon$ are respectively $0.09, 0.1296, 11/6, 1/1.4245$ and $P = S : \nabla u$.

The constant c_2 and c_ε are different from their original values of 1.92 and 1/1.3.

The constant c_2 is adjusted to reproduce the decay of k in isotropic turbulence. With $u = 0, \rho = \rho_\infty, T = T_\infty$ the model gives

$$k = k_0(1 + (c_2 - 1)\frac{\varepsilon_0}{k_0}t)^{\frac{-1}{c_2 - 1}},$$

The experimental results of Comte-Bellot [4] give a decay of k in $t^{-1.2}$ and this fixes $c_2 = 11/6$ while $c_2 = 1.92$ leads to a decay in $t^{-1.087}$ and therefore to an overestimation of k.

This has also been reported in [5], where the author managed to compute the right recirculating bubble length for the backward step problem using the standard $k - \varepsilon$ model with this new value $c_2 = 11/6$, wall-laws and $c_\varepsilon = 1/1.3$.

Finally, the compatibility relation between the $k-\varepsilon$ constants which comes from the requirement of a logarithmic velocity profile in the boundary layer [1] gives the c_ε constant:

$$c_\varepsilon = \frac{1}{\kappa^2\sqrt{c_\mu}}(c_2 c_\mu - c_1) = \frac{1}{1.423}, \quad \kappa = 0.41,$$

to be compared to the classical value of $c_\varepsilon = 1/1.3$.

7.1 Boundary and Initial Conditions

The previous system of Navier-Stokes and $k-\varepsilon$ equations is well posed[1] with the following set of boundary conditions:

Inflow and outflow :

The idea is to avoid boundary layers such that all second order derivatives are removed and that the remaining system (Euler-$k-\varepsilon$ model) is a system of conservation laws not anymore coupled (as we dropped the turbulent contributions to first order derivative terms). Inflow and outflow boundary conditions are of characteristic types. Where roughly the idea is to impose the value of a variable if the corresponding wave is entering the domain following the sign of the corresponding eigenvalue (in 3D):

$$\lambda_1 = u.n + c, \lambda_{2,3,4} = u.n, \lambda_5 = u.n - c, \lambda_{6,7} = u.n,$$

where n is the unit outward normal. However, as the system cannot be fully diagonalized, we use the following approach [10]. Along these boundaries the fluxes are split in positive and negative parts following the sign of the eigenvalues of the Jacobian A of the convective operator F.

$$\int_{\Gamma_\infty} F.n d\sigma = \int_{\Gamma_\infty} (A^+ W_{in} + A^- W_\infty).n d\sigma,$$

where W_{in} is the computed (internal) value at the previous iteration and W_∞ the external value, given by the flow.

Symmetry :

Here again the idea is to avoid boundary layers. We drop terms with second order derivatives and the slipping boundary condition ($u.n = 0$) is imposed in weak form.

Solid walls :

The physical boundary condition is a no-slip boundary condition for the velocity ($u = 0$) and for the temperature, either an adiabatic condition ($\partial_n T = 0$) or an isothermal condition ($T = T_\Gamma$). However, the $k-\varepsilon$ model above is not

[1] In the small as the mathematicians say, meaning that the solution exists for a small time interval at least.

valid [1] near walls because the turbulence is not isotropic at small scales. In the wall-laws approach the near-wall region is removed from the computational domain and the previous conditions are replaced by by Fourier conditions of the type

$$u_\delta.n = 0, \quad u_\delta.t = f_1(\partial_n u_\delta, \partial_n T_\delta), \quad T_\delta = f_2(\partial_n u_\delta, \partial_n T_\delta)$$

for isothermal walls. This will be described in more details later.

Initial conditions :

For external as well as internal flows, the initial flow is taken uniform with small values for k_0 and ε_0 (basicly $10^{-6}|u_0|$). We take the same value for k and ε leading to a large turbulent time scale $k/\varepsilon = 1$ which characterizes a laminar flow.

$$u = u_0, \quad \rho = \rho_0, \quad T = T_0, \quad k = k_0, \quad \varepsilon = \varepsilon_0.$$

Internal flow simulations often also require given profiles for some quantities. This is prescribed on the corresponding boundaries during the simulation.

8 Wall-Laws

The general idea in wall-laws is to remove the stiff part from boundary layers, replacing the classical no-slip boundary condition by a more sophisticated relation between the variables and their derivatives. We introduce the a constant quantity called friction velocity from:

$$\rho_w u_\tau^2 = (\mu + \mu_t)\frac{\partial u}{\partial y}|_{y=\delta} = \mu\frac{\partial u}{\partial y}|_{y=0}, \tag{36}$$

where w means at walls and δ at a distance δ from the real wall. Using u_τ we introduce a local Reynolds number:

$$y^+ = \frac{\rho_w y u_\tau}{\mu_w}. \tag{37}$$

The aim is now to express the behavior of $u^+ = u/u_\tau$ in term of y^+ which means that the analysis will be independent of the Reynolds number.

In this section we describe our approach to wall-laws. We also give an extension to high speed separated flows with adiabatic and isothermal walls. The ingredients are:

1. Global wall-laws: numerically valid up to the wall (i.e. $\forall y^+ \geq 0$).

2. Weak formulation: pressure effects are taken into account in the boundary integrals which come for the integrations by parts.

3. Small δ in wall-laws: this means that the computational domain should not be too far from the wall.

4. Fine meshes: in the sense that the computational mesh should be fine enough for the numerical results to become mesh independent.

5. Compressible extension: laws valid for a large range of Mach number.

An important and interesting feature of wall-laws is that they are compatible with explicit time scheme,

something which is not so real with low-Reynolds corrections.

8.1 Generalized wall functions for u

The first level in the modeling for wall-laws is to consider flows attached (i.e. without separations) on adiabatic walls (i.e . $\frac{\partial T}{\partial n} = 0$). We are looking for laws valid up to the wall (i.e. valid $\forall y^+$). We consider the following approximated momentum equation in near-wall regions (x and y denote the local tangential and normal directions):

$$\partial_y((\mu + \mu_t)\partial_y u) = 0, \tag{38}$$

where

$$\mu_t = \kappa\sqrt{\rho\rho_w} y u_\tau (1 - e^{-y^+/70}), \quad \text{with } y^+ = \frac{\rho_w u_\tau y}{\mu_w}, \tag{39}$$

is a classical expression for the eddy viscosity valid up to the wall. (38) means that the shear stress along y is constant. u_τ is a constant called the friction velocity and is defined by :

$$u_\tau = (\frac{(\mu + \mu_t)}{\rho_w}\frac{\partial u}{\partial y})^{1/2} = \text{Constant}, \tag{40}$$

subscript w means at the wall.

8.1.1 High-Reynolds regions

In high Reynolds regions the eddy viscosity $\mu_t = \kappa\sqrt{\rho\rho_w} y u_\tau$ dominates the laminar one and this leads to the log law

$$\frac{\partial u}{\partial y} = \sqrt{\frac{\rho_w}{\rho}}\frac{u_\tau}{\kappa y}, \quad u = u_\tau\sqrt{\frac{\rho_w}{\rho}}(\frac{1}{\kappa}log(y) + C),$$

provided that $\frac{\partial \rho}{\partial y} << \frac{\partial u}{\partial y}$ which is acceptable because $\frac{\partial p}{\partial y} \sim 0$ and $\frac{\partial T}{\partial y} = 0$ as the wall is adiabatic. Therefore $\frac{\partial \rho}{\partial y} \sim 0$.

We can see that at this level, there is no explicit presence of the Reynolds number. The dependency with respect to the Reynolds number is in the constant C. To have an universal expression, we write,

$$u = u_\tau\sqrt{\frac{\rho_w}{\rho}}(\frac{1}{\kappa}\log(\frac{y u_\tau \rho_w}{\mu_w}) + \beta),$$

where $\beta = -\log(u_\tau \rho_w/\mu_w) + C)$ is found to have a universal value of about 5 for incompressible flows [8]:

$$u^+ = \frac{u}{u_\tau} = \sqrt{\frac{\rho_w}{\rho}}(\frac{1}{\kappa}log(y^+) + 5).$$

Note that we always use wall values to reduce y-dependency as much as possible. This is important for numerical implementation.

8.1.2 Low-Reynolds regions

In low-Reynolds regions, (39) is negligible and (38) gives a linear behavior for u vanishing at walls:

$$\rho_w u_\tau^2 = \mu\frac{\partial u}{\partial y} \sim \mu\frac{u}{y}.$$

In other words, we have:

$$u^+ = \frac{u}{u_\tau} = \frac{y u_\tau \rho_w}{\mu_w} = y^+.$$

8.1.3 General expression

To have a general expression, we define the friction velocity u_τ as solution of

$$u = u_\tau\sqrt{\frac{\rho_w}{\rho}}f(u_\tau).$$

where f is such that $w = u_\tau\sqrt{\frac{\rho_w}{\rho}}f(u_\tau)$ is solution of (38-39). The wall-function therefore is not known explicitly and depends on density distribution. A hierarchies of laws can be obtained therefore by taking into account compressibility effects starting from low-speed laws (see appendix 1). Our aim during this development is to provide laws easy to implement for unstructured meshes. For low-speed flows, where density variations are supposed negligible, a satisfactory choice for f is the nonlinear Reichardt function f_r defined by:

$$f_r(y^+) = 2.5\log(1 + \kappa y^+) + 7.8(1 - e^{-y^+/11} - \frac{y^+}{11}e^{-0.33y^+}), \qquad (41)$$

This expression fits both the linear and logarithmic velocity profiles and also its behavior in the buffer region.

8.2 Wall function for T - energy equation

Consider the viscous part of the energy equation written in the boundary layer (i.e. $\partial_x << \partial_y$):

$$\frac{\partial}{\partial y}(u(\mu + \mu_t)\frac{\partial u}{\partial y}) + \frac{\partial}{\partial y}((\chi + \chi_t)\frac{\partial T}{\partial y}) = 0.$$

When we integrate this equation between the fictitious wall ($y = \delta$) and the real one ($y = 0$), we obtain :

$$(\chi + \chi_t)\frac{\partial T}{\partial y} \mid_\delta - \chi\frac{\partial T}{\partial y} \mid_0 = u(\mu + \mu_t)\frac{\partial u}{\partial y} \mid_0 - u(\mu + \mu_t)\frac{\partial u}{\partial y} \mid_\delta . \qquad (42)$$

So, thanks to $\frac{\partial T}{\partial y} \mid_0 = 0$ and $u \mid_0 = 0$:

$$(\chi + \chi_t)\frac{\partial T}{\partial y} \mid_\delta + u(\mu + \mu_t)\frac{\partial u}{\partial y} \mid_\delta = 0. \qquad (43)$$

Therefore, in the adiabatic case, there is no term for the energy equation to account for [11] while for isothermal walls we need to close (44) providing an expression for either the first term in the left hand side or for the right hand side (see appendix 2):

$$(\chi + \chi_t)\frac{\partial T}{\partial y} \mid_\delta + u(\mu + \mu_t)\frac{\partial u}{\partial y} \mid_\delta = \chi\frac{\partial T}{\partial y} \mid_0 . \qquad (44)$$

Remark: As a consequence, to evaluate the heat transfer at the wall, we have to use the following formula:

$$C_h = \frac{\chi \partial_y T \mid_0}{\rho_\infty u_\infty^3 \gamma} = \frac{(\chi + \chi_t)\partial_y T \mid_\delta + u\rho_w u_\tau^2}{\rho_\infty u_\infty^3 \gamma}.$$

This is important as industrial codes usually do the post-processing in a separate level than computation and the fluxes are not communicated between the two modules. In other words, with these codes, when using wall functions as well as with low-Reynolds models, only the first term is present in a heat flux evaluation above. This might also explain some of the reported weakness of wall functions for heat transfer prediction.

Remark:

In separation and recirculation areas u and u_τ needed by our wall-laws are small. As a consequence, this leads to an underestimation of the heat flux. In these area, by dimension argument, we choose the local velocity scale to be:

$$u = c_\mu^{-3/4}\sqrt{k}.$$

And redefine, the friction flux by:

$$(\mu + \mu_t)\frac{\partial u}{\partial y} = c_\mu^{-3/4}(\mu + \mu_t)\frac{\partial \sqrt{k}}{\partial y} . \qquad (45)$$

8.3 k and ε

Once u_τ is computed, k and ε are set to:

$$k = \frac{u_\tau^2}{\sqrt{c_\mu}}\alpha\,, \quad \varepsilon = \frac{k^{\frac{3}{2}}}{l_\varepsilon}\,, \tag{46}$$

where $\alpha = min(1, (\frac{y^+}{20})^2)$ reproduces the behavior of k when δ tends to zero (δ is the distance of the fictitious computational domain from the solid wall). The distance δ is given a priori and is kept constant during the computation. l_ε is a length scale containing the damping effects in the near wall regions.

$$l_\varepsilon = \kappa c_\mu^{-3/4} y(1 - e^{\frac{-y^+}{2\kappa c_\mu^{-3/4}}}).$$

Again, here the limitation is for separation points where the friction velocity goes to zero while high level for k would be expected.

9 Appendix 1: Generalized wall functions

We give here a hierarchies of development accounting for various aspect of local flow behavior on the fictitious wall. This leads to general wall functions valid for separated and unsteady flows as well as flows with compressibility effects.

9.1 Pressure correction

This is an attempt to take into account the pressure gradient and convection effects in the classical wall-laws.

To account for pressure and convection effects, f_c is added to Reichardt's equation in (8.1.3):

$$f(u_\tau) = f_r(u_\tau) + f_c(u_\tau).$$

f_c is a new contribution when pressure and convection effects exist.

The simplified momentum equation (38) is enhanced to

$$\partial_y((\mu + \mu_t)\partial_y u) = \partial_x p + \partial_x(\rho u^2) + \partial_y(\rho uv) \tag{47}$$

where μ_t is given by (39). Suppose that the right hand side of equation (47) is known and constant close to the wall:

$$C = \partial_x p + \partial_x(\rho u^2) + \partial_y(\rho uv).$$

We can then integrate this equation in y, not exactly unfortunately, but only after a first order development in y of the exponential near $y = 0$. The equation is written in terms of y^+.

$$\partial_{y^+}((1 + \kappa y^+(1 - e^{y^+/70}))\partial_{y^+}u) = C\mu u_\tau^{-2} = C',$$

So,

$$\partial_{y^+}((1 + {y^+}^2\frac{\kappa}{70})\partial_{y^+}) = C'.$$

Hence,

$$\partial_{y^+}u = \frac{C'y^+}{1 + \frac{\kappa}{70}{y^+}^2} + \frac{A}{1 + \frac{\kappa}{70}{y^+}^2}.$$

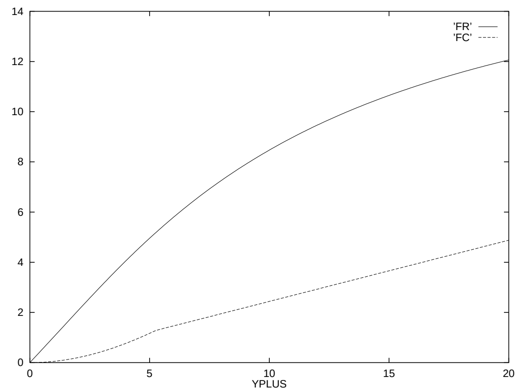

Figure 1: Reichardt law and the tangential correction for a given pressure gradient for $y^+ < 20$.

After a second integration and using the boundary conditions (i.e. at the wall and at δ) we have:

$$u = \frac{35C\mu}{\kappa u_\tau^2}\log(1+\frac{\kappa}{70}(y^+)^2). \tag{48}$$

On the other hand, the equation is also easily integrated when $\mu_t = \kappa\rho y u_\tau$. Hence the corrections are respectively

$$f_c(y^+) = (\frac{35C\nu}{\kappa u_\tau^3})\log(1+\frac{\kappa}{70}(y^+)^2) \quad \text{when} \quad y^+ << 70, \tag{49}$$

and

$$f_c(y^+) = \frac{C\delta}{\kappa u_\tau^2} \quad \text{elsewhere}. \tag{50}$$

These expressions intersect at $y^+ = 5.26$. This is in contradiction with the limit $y^+ = 70$ given by (39). However, as we would like the correction to perturb weakly the Reichardt law, we define the correction as the minimum of the two expressions. Hence, we apply (49) for $y^+ \leq 5.26$ and (50) for $y^+ \geq 5.26$. Of course, this correction vanishes with C and we recover the Reichardt law.

9.2 Corrections on adiabatic walls for compressible flows

Note that the previous wall-laws are valid for incompressible flows. We need to introduce therefore some corrections to take into account the compressible feature of the flow. By now, ∞ will denote reference quantities and e will refer to the nearest local value outside boundary layer. We need to account for density variation in (8.1.3) and for the fact that Reichardt law has been suggested for low-speed flows.

9.3 Prescribing ρ_w

Let us define the recovery factor [8]:

$$r = \frac{T_f - T_e}{T_{ie} - T_e},$$

where T_f is called the friction temperature and T_{ie} is given by :

$$T_{ie} = T_e\,(1 + \frac{\gamma - 1}{2} M_e^2).$$

For turbulent flows, it is admitted that $r = Pr^{1/3}$ [8]. We obtain

$$T_f = T_e\,(1 + Pr^{\frac{1}{3}} \frac{\gamma - 1}{2} M_e^2).$$

In the adiabatic case, the wall temperature is the friction temperature T_f [8] (i.e. $T_w = T_f$).

To solve (38-39) in the adiabatic compressible case, we have to provide μ_w and ρ_w. The viscosity at the wall μ_w is obtained from the Sutherland law:

$$\mu_w = \mu_e\,(\frac{T_w}{T_e})^{1/2}\,\frac{1 + 110.4/T_e}{1 + 110.4/T_w}\,. \tag{51}$$

For the second quantity, we use the Crocco relation [8]:

$$T = T_w + (T_{ie} - Tw)\,\frac{u}{u_e} - (T_{ie} - T_e)\,(\frac{u}{u_\infty})^2\,,$$

As a consequence, we have :

$$\frac{T}{T_w} = 1 + [(1 + \frac{\gamma - 1}{2} M_e^2) \frac{T_e}{T_w} - 1] \frac{u}{u_e} - \frac{\gamma - 1}{2} M_e^2 \frac{T_e}{T_w} \left(\frac{u}{u_e}\right)^2 .$$

We suppose the static pressure constant in the normal direction (i.e. $\partial_y p = 0$), therefore, from the perfect gas law, we obtain :

$$\frac{\rho_w}{\rho} = \frac{T}{T_w}\,.$$

As a consequence, we evaluate ρ_w thanks to :

$$\rho_w\; = \rho[\,1\; +\; [(1 + \frac{\gamma - 1}{2} M_e^2) \frac{T_e}{T_w} - 1]\,\frac{u}{u_e}\; - \frac{\gamma - 1}{2} M_e^2 \frac{T_e}{T_w}\,\left(\frac{u}{u_e}\right)^2]. \tag{52}$$

However, the numerical implementation of (51) and (52) are not straightforward as we do not know the e denoted values (i.e. u_e, T_e, M_e) on unstructured meshes.

We therefore choose to use only quantities known for any unstructured meshes: at the 'fictitious' wall or at inflow. In particular, M_e is replaced by $M_\delta = \sqrt{\frac{u^2+v^2}{\frac{\gamma P}{\rho}}}$. More precisely, knowing $(\rho_\delta, T_\delta, M_\delta, u_\infty, T_w)$ we find ρ_w by:

$$\rho_w\; = \rho_\delta[\,1\; +\; [(1 + \frac{\gamma - 1}{2} M_\delta^2) \frac{T_\delta}{T_w} - 1]\,\frac{u_\delta}{u_\infty}\; - \frac{\gamma - 1}{2} M_\delta^2 \frac{T_\delta}{T_w}\,\left(\frac{u_\delta}{u_\infty}\right)^2]. \tag{53}$$

9.4 Correction for Reichardt law

The next step is to introduce a correction for the Reichardt law. We can use one of the following approaches.

9.4.1 Using mixing length formula

Following Cousteix [8], express the turbulent tension thanks to the mixing-length formula for high-Reynolds region ($\kappa y \partial_y u = u_\tau$):

$$\rho_w u_\tau^2 = \rho \kappa^2 y^2 (\partial_y u)^2 ,$$

so that:

$$\partial_y u = \sqrt{\frac{\rho_w}{\rho}} \frac{u_\tau}{\kappa y}.$$

Express ρ_w / ρ thanks to the Crocco law and obtain:

$$\frac{\partial u}{\partial y} = \frac{u_\tau}{\kappa y} \left(1 + b \frac{u}{u_\infty} - a^2 \left(\frac{u}{u_\infty} \right)^2 \right) , \tag{54}$$

with $a^2 = \frac{\gamma - 1}{2} M_\delta^2 \frac{T_\delta}{T_w}$ and $b = (1 + \frac{\gamma - 1}{2} M_\delta^2) \frac{T_\delta}{T_w} - 1$. The weakness of this approach is that it is not valid up to the wall. A global correction needs a global mixing length formula as starting point using (8.1.3):

$$\partial_y u = u_\tau (\partial_y f(y^+) \sqrt{\frac{\rho_w}{\rho}} + f(y^+) \partial_y (\sqrt{\frac{\rho_w}{\rho}})), \tag{55}$$

which is hardly computable.

Remark:

A similar approach comes from Van Driest [8, 9], with the log law relation as starting point in (57). Unfortunately, this also leads to a relation only valid for high Reynolds region.

9.4.2 Restart from (38-39)

To avoid the difficulty above, we would like to restart from our boundary layer system for u:

$$\left(\mu + \sqrt{\rho \rho_w} \kappa u_\tau y (1 - e^{-y^+/70}) \right) \frac{\partial u}{\partial y} = \rho_w u_\tau^2. \tag{56}$$

Now, suppose that the Reichardt law is obtained after integration of:

$$u_\tau = \partial_y u \left(\mu + \kappa y (1 - e^{-y^+/70}) \right) . \tag{57}$$

First consider the case $y^+ > 100$ and drop the laminar viscosity. Hence, replacing u_τ by (57) in the left hand side of (56), leads to:

$$\frac{\partial u}{\partial y} = \sqrt[4]{\frac{\rho_w}{\rho}} \frac{u_\tau}{\kappa y (1 - e^{-y^+/70})}. \tag{58}$$

The Crocco law links density and temperature and (58) becomes:

$$\frac{\partial u}{\partial y} = \sqrt[4]{1 + b \frac{u}{u_\infty} - a^2 \left(\frac{u}{u_\infty} \right)^2} \frac{u_\tau}{\kappa y (1 - e^{-y^+/70})} , \tag{59}$$

The integration of relation (59) is not possible. At this level, we use the following approximation:

$$\left(\frac{1}{a}(\arcsin\frac{2\,a^2\,u/u_\infty\,-\,b}{\sqrt{b^2+4\,a^2}}+\arcsin\frac{b}{(b^2+4\,a^2)^{1/2}})\right)^{1/4}=u_\tau\,f_r(y^+).$$

If laminar viscosity dominates the eddy one ($y^+ < 5$),

$$\rho_w\,u_\tau^2=\mu\partial_y u. \tag{60}$$

For $5 < y^+ < 100$, we use a linear interpolation between the two expressions above.

Numerical experiences have shown better behavior for this second approach.

10 Appendix 2: wall functions for isothermal walls

For isothermal walls ($T_w = T_{given}$), we have to provide a law for the temperature, as we did for the velocity. In weak form, we only need a law for the thermal stress $\chi\frac{\partial T}{\partial y}$.

10.0.3 The Reynolds relation

A first attempt to evaluate $\chi\frac{\partial T}{\partial y}$ is to use the classical Reynolds relation between heat and friction coefficients [8]:

$$C_h=\frac{s\,C_f}{2}=\frac{1.24}{2}C_f=0.62C_f. \tag{61}$$

So, we have :

$$-\frac{\chi\partial_y T}{\rho u^3\gamma}=C_h=0.62C_f=1.24\frac{\rho_w\,u_\tau^2}{\rho u^2},$$

Note that, for the isothermal case, as $(\mu+\mu_t)\partial_y u=\rho_w\,u_\tau^2$, (42) leads to :

$$(\chi+\chi_t)\partial_y T\mid_\delta+u\rho_w\,u_\tau^2=\chi\partial_y T.$$

So, we have [11] :

$$(\chi+\chi_t)\partial_y T\mid_\delta+u\rho_w\,u_\tau^2=-1.24\rho_w\,u_\tau^2\gamma u. \tag{62}$$

Again, in the definition of the friction and heat coefficients, we use local values instead of reference ones. The wall density is obtained through the Crocco's law (53).

10.0.4 Crocco's method

A more general approach consists in linearizing the local Crocco law for the temperature to obtain the normal temperature slope. We have :

$$\frac{T}{T_w}\;=\;1\;+\;[(1+\frac{\gamma-1}{2}M^2)\frac{T}{T_w}-1]\;u\;-\;\frac{\gamma-1}{2}M^2\frac{T}{T_w}\;u^2\,.$$

We obtain :

$$\begin{aligned}\frac{\partial T}{\partial y}=&(\frac{\gamma-1}{2}2M\frac{\partial M}{\partial y}T+(1+\frac{\gamma-1}{2}M^2)\frac{\partial T}{\partial y})\;u\\&+[(1+\frac{\gamma-1}{2}M^2)T-T_w])\frac{\partial u}{\partial y}\\&-\frac{\gamma-1}{2}(2MT\frac{\partial M}{\partial y}+M^2\frac{\partial T}{\partial y})u^2-\frac{\gamma-1}{2}M^2T\,2u\frac{\partial u}{\partial y}\,.\end{aligned} \tag{63}$$

Moreover, by definition, $M^2 = u^2\rho/\gamma p$. As a consequence :

$$\frac{\partial M}{\partial T} = \sqrt{\frac{\rho}{\gamma p}}\frac{\partial u}{\partial y} + \frac{u}{\sqrt{\gamma p}}\frac{1}{2\sqrt{\rho}}\frac{\partial \rho}{\partial y}$$

because the static pressure is supposed to be constant in the normal direction.
Finally, we have to express $\partial_y \rho$. Thanks to the perfect gas law and as the static pressure is constant in the normal direction, we find :

$$\frac{\partial p}{\partial y} = 0 = (\gamma - 1)(\rho\frac{\partial T}{\partial y} + T\frac{\partial \rho}{\partial y}),$$

so :

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{T}\frac{\partial T}{\partial y}.$$

As a consequence, we have :

$$\frac{\partial M}{\partial T} = \sqrt{\frac{\rho}{\gamma p}}\frac{\partial u}{\partial y} - \frac{M}{2T}\frac{\partial T}{\partial y}.$$

Reported in (63), we find:

$$\frac{\partial T}{\partial y} = \frac{1.5(\gamma-1)M^2T + T - T_w - 2(\gamma-1)M^2Tu}{1-u}\frac{\partial u}{\partial y}$$

where, $\partial_y u$ is given by(36):

$$\frac{\partial u}{\partial y} = \frac{\rho_w u_\tau^2}{\mu + \mu_t}.$$

At last, The temperature slope prescribed by Crocco law is:

$$(\chi + \chi_t)\frac{\partial T}{\partial y}\mid_\delta = \frac{\chi + \chi_t}{\mu + \mu_t}\frac{(\gamma-1)M^2T(1.5 - 2u) + T - T_w}{1-u}\rho_w u_\tau^2 \tag{64}$$

where all the values are local ones. This expression is valid up to the wall and can be used for $\delta = 0$.

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