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# Shape Sensitivity and Design for Fluids with Shocks

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Dedicated to Professor Mutsuto Kawahara on the occasion of his 60th birthday

For many problems of compressible fluid dynamics it is desirable to find the sensitivity of the shock position with respect to the shape of the domain occupied by the fluid. One application is for the minimization of the sonic boom of airplanes; another is for the stability of the stream in fast-flowing rivers or canals. Classical calculus of variation is not valid for these cases because of the presence of Dirac functions appearing when a discontinuous function is differentiated, but we show here on the compressible potential flow equation how to find the equations of the derivatives and what are the linearized problems. Some numerical test cases are given for illustration.

*Keywords*: Partial differential equations; Nozzle flow; Sensitivity; Transonic equation; Shallow water equations; Optimal shape design

## INTRODUCTION

There are two pioneering papers on the linearized Euler equations for potential flow: one by Majda (1983) and one by Godlewski *et al.* (1998). There are also many studies directed to the Euler equation (see Giles and Pierce, 2001; Alekseev and Navon, 2002; Homescu and Navon, 2003 for example). In Bardos and Pironneau (2002a,b) we have reinterpreted the results of Godlewski *et al.* (1998) and introduced a formalism which we shall apply here to find the linearized transonic equation in cases where the position of the shock is sensitive to the parameter of the linearization.

Following di Cesare and Pironneau (2000) and extending Pironneau (2002) we analyze the stationary problem for the transonic equation, we formally derive an equation for the derivative of the potential of the flow with respect to parameters in the data, including the shape of the domain, we discuss the well posedness of the problem and we present some numerical tests to confirm the theory. This is done first in one dimension of space for stationary and transient cases and then for two-dimensional cases.

We also adapt the results to the shallow water equations for the stability of streams in rivers and canals.

## COMPRESSIBLE POTENTIAL FLOWS

### Isentropic Flow

Consider the Euler equations for compressible perfect isentropic flows in a domain $\Omega$ of boundary $\Gamma$.

$$\partial_t\rho+\nabla\cdot(\rho u)=0,\quad \partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla\rho^\gamma=0. \quad (1)$$

For aerodynamics $\gamma=1.4$ and for shallow waters $\gamma=1$.

Initial and boundary conditions must be given such as $\rho=\rho^0$, $u=u^0$ at time zero in $\Omega$ and $u\cdot n=u_\Gamma\cdot n$ on $\Gamma$ plus $\rho=\rho_\Gamma$, on $\Gamma^-=\{x\in\Gamma : u_\Gamma\cdot n<0\}$. If in addition the inflow velocity is supercritical then $u\cdot s=u_\Gamma\cdot s$ must be given too. The vectors $n$, $s$ are the outer normal and tangent vector(s) to $\Gamma$.

### Irrotational Flow

The following holds for any vector field $u$

$$\nabla\cdot(\rho u\otimes u)=\rho u\cdot\nabla u+\nabla\cdot(\rho u)u$$

$$=\rho\nabla\frac{u^2}{2}-\rho u\times\nabla\times u+\nabla\cdot(\rho u)u. \quad (2)$$

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***The Case* $\gamma > 1$**

So the irrotational stationary solutions to Eq. (1) satisfy (recall that $\nabla\rho^\gamma = (\gamma\rho/\gamma-1)\nabla\rho^{\gamma-1}$):

$$\nabla\times u = 0 \quad \partial_t\rho + \nabla\cdot(\rho u) = 0,$$
$$\partial_t u + \nabla\left(\frac{u^2}{2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right) = 0.$$

The first equation tells us that $u$ derives from a potential $\phi$, i.e. $u = \nabla\phi$. The third equation gives an algebraic relation between $\rho$ and $u^2$, $2\gamma\rho^{\gamma-1} + (\gamma-1)u^2 = K - \partial_t\phi$, where the constant $K$ is fixed by the boundary conditions. After renormalization it can be written as

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \rho^{1/\beta} = 1 \quad \partial_t\rho + \nabla\cdot(\rho\nabla\phi) = 0. \tag{3}$$

Stationary solutions of this system must satisfy the *transonic equation*:

$$\begin{aligned}
&\nabla\cdot\left(\left(1-|\nabla\phi|^2\right)^\beta\nabla\phi\right) = 0 \text{ in } \Omega,\\
&\left(1-|\nabla\phi|^2\right)^\beta\frac{\partial\phi}{\partial n}|_{\Gamma_N} = g,\\
&\phi = \phi_\Gamma \text{ on } \Gamma_D := \Gamma\backslash\Gamma_N,
\end{aligned}\tag{4}$$

with $\gamma = 1.4$, $\beta = 1/(\gamma-1) = 2.5$ in air. Boundary conditions come from the knowledge of $\rho u\cdot n$ on $\Gamma$. On $\Gamma^-$, $\partial\phi/\partial s$ is also known, or equivalently $\phi$ up to a constant $k_j$ on each connected component $\Gamma_j^-$ of $\Gamma^-$.

There are also integral quantities which are prescribed and which come from an integration over time of Eq. (1), the conservation of mass and the conservation of momentum which may put a constraint of compatibility on the data above:

$$\int_\Omega\rho + \int_\Gamma\rho u\cdot n = \int_\Omega\rho^0 + \int_\Gamma\rho^0u^0\cdot n,$$
$$\int_\Omega\rho u + \int_\Gamma\rho u(u\cdot n) = \int_\Omega\rho^0u^0 + \int_\Gamma\rho^0u^0(u^0\cdot n).$$

For instance, from Eq. (4) we see that it is necessary that the integral on $\Gamma$ of $g$ be zero, therefore the integral of $\rho$ on $\Omega$ is prescribed.

An entropy inequality must be added for well posedness (see Glowinski, 1984; Nečas, 1989):

$$\Delta\phi > -\infty.$$

It is automatically satisfied when $u$ is continuous and also when $u$ is discontinuous with a decreasing jump in the direction of the flow $u$.

A time-dependent, simple (yet physically meaningful) model between Eqs. (3) and (4) is

$$\begin{aligned}
&\partial_{tt}\phi - \nabla\cdot\left(\left(1-|\nabla\phi|^2\right)^\beta\nabla\phi\right) = 0 \text{ in } \Omega\times(0,T)\\
&\phi = \phi^0,\ \partial_t\phi = \phi^1 \text{ on } \Omega\times\{0\}\\
&\left(1-|\nabla\phi|^2\right)^\beta\frac{\partial\phi}{\partial n} = g \text{ on } \Gamma_N\times(0,T),\\
&\phi = \phi_\Gamma \text{ on } \Gamma_D\times(0,T).
\end{aligned}\tag{5}$$

***The Case* $\gamma = 1$**

For shallow water flows, $\gamma = 1$ and the trick used, namely

$$\nabla\rho^\gamma = \frac{\gamma}{\gamma-1}\rho\nabla\rho^{\gamma-1}$$

must be replaced by $\nabla\rho = \rho\nabla\log\rho$. Then Eq. (3) becomes

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \log\rho = 1 \quad \partial_t\rho + \nabla\cdot(\rho\nabla\phi) = 0, \tag{6}$$

and the stationary *potential flow equation* for shallow waters is:

$$\begin{aligned}
&\nabla\cdot\left(e^{-\frac{1}{2}|\nabla\phi|^2}\nabla\phi\right) = 0 \text{ in } \Omega,\\
&e^{-\frac{1}{2}|\nabla\phi|^2}\frac{\partial\phi}{\partial n}|_{\Gamma_N} = g, \quad \phi|_{\Gamma_D} = \phi_\Gamma.
\end{aligned}\tag{7}$$

The time-dependent approximation of Eq. (6) is

$$\begin{aligned}
&\partial_{tt}\phi - \nabla\cdot\left(e^{-\frac{1}{2}|\nabla\phi|^2}\nabla\phi\right) = 0 \text{ in } \Omega\times(0,T)\\
&\phi = \phi^0, \quad \partial_t\phi = \phi^1 \text{ on } \Omega\times\{0\}\\
&e^{-\frac{1}{2}|\nabla\phi|^2}\frac{\partial\phi}{\partial n} = g \text{ on } \Gamma_N\times(0,T),\\
&\phi = \phi_\Gamma \text{ on } \Gamma_D\times(0,T).
\end{aligned}\tag{8}$$

**Slender Shock Tube**

We can compound the case $\gamma = 1$ with the case $\gamma > 1$ in one framework by defining

$$\rho(u) = (1-u^2)^\beta \text{ with } \beta = \frac{1}{\gamma-1} \text{ if } \gamma > 1 \text{ and}$$

$$\rho(u) = e^{-\frac{1}{2}u^2} \text{ if } \gamma = 1.$$

When $\Omega$ is slender of length $L$ on the $x$ axis with cross-section of surface $S(x)$ then, following Landau and Lifschitz (1956), the solution to Eq. (4) is

$$\rho(u)u(x)S(x) = K, \quad \int_0^L\rho(u)S(x)\,\mathrm{d}x = a \tag{9}$$

where $K$ is fixed by the inflow/outflow boundary conditions and $a$ is the prescribed total mass of gas. Equation (9a) has two solutions at each $x$, one subcritical and one supercritical. Only one discontinuous switch from super to subcritical is allowed by the entropy condition and the position of the switch (shock) is determined by Eq. (9b).

### *Derivative with Respect to the Total Mass*

Let $u'$ denote the derivative of $u$ with respect to $a$. By differentiation of Eq. (9a) we find that $u' = 0$ almost everywhere. However, the position of the shock $x_s$ varies with $a$ and since

$$u = u_- + (u_+ - u_-)H(x - x_s),$$

where $H$ is the Heavyside function, we have

$$u' = -x_s'[u]\delta(x - x_s) \text{ where } [u] := u_+ - u_-.$$

By the same principle

$$\frac{\mathrm{d}}{\mathrm{d}a}\rho(u) = -x_s'[\rho(u)]\delta(x - x_s).$$

Putting this information into Eq. (9b) differentiated gives the shock displacement:

$$x_s' = -\frac{1}{[\rho(u)]S(x_s)}.$$

### *Derivative with Respect to the Inflow/Outflow Condition*

The derivative $u'$ of $u$ with respect to $K$ will have two parts, a regular part denoted by $u_K'$ and a singular part which as before is $-x_s'[u]\delta(x - x_s)$. After differentiation, Eq. (9) is

$$M(u)u_K' = \frac{1}{S} \text{ with } M(u) = \left| \begin{array}{ll} (1-u^2)^\beta\left(1 - 2\beta\frac{u^2}{1-u^2}\right) & \text{if } \gamma > 1 \\ (1-u^2)e^{-\frac{1}{2}u^2} & \text{if } \gamma = 1 \end{array} \right. \tag{10}$$

$$[\rho(u)]S(x_s)x_s' = -\int_0^L \dot\rho(u)u_K'S\,\mathrm{d}x \text{ where}$$

$$\dot\rho(u) := \frac{\mathrm{d}\rho(u)}{\mathrm{d}u} \text{ giving } x_s' = \frac{-1}{[\rho(u)]S(x_s)}\int_0^L \frac{\dot\rho(u)}{M(u)}\mathrm{d}x.$$

### *Derivative with Respect to Shape*

Assume now that $S(\cdot)$ depends on a parameter $b$ and denote $S'$ its derivative in $b$. Differentiation of Eq. (9) yields

$$M(u)u_S' = -\frac{KS'}{S^2},$$

$$[\rho(u)]S(x_s)x_s' = \int_0^L (\dot\rho(u)u_S'S + \rho(u)S')\mathrm{d}x. \tag{11}$$

Here too $x_s'$ can be computed explicitly:

$$x_s' = -\frac{1}{[\rho(u)]S(x_s)}\int_0^L\left(\frac{K\dot\rho(u)}{M(u)S} - \rho(u)\right)S'\mathrm{d}x.$$

*Remark 1* In all cases the shock position is unstable when $[u] \to 0$ (weak shocks).

### The Time-dependent Slender Case

Integrated over the cross-section of the slender domain $\{(x, y) : -S_m(x) < y < S_M(x),\ x \in (0, L)\}$, with $S = S_M - S_m$, Eq. (5) leads to

$$\partial_{tt}\phi - \frac{1}{S}\partial_x(S\rho(\partial_x\phi)\partial_x\phi) = 0. \tag{12}$$

As before when $S$ depends on a parameter $a$, the derivative $\phi'$ with respect to $a$ will be discontinuous and will satisfy in the sense of distribution theory:

$$\partial_{tt}\phi' - \partial_x(\partial_u\rho(u)\partial_x\phi + \rho(u)\partial_x\phi') - \frac{\partial_x S}{S}M(u)u' = \left(\frac{\partial_x S}{S}\right)'\rho(u)u. \tag{13}$$

As an illustration assume that $S'$ is zero before the shock and that stability is studied at a stationary equilibrium state. Then $\rho(u)u = K$ before the shock and so $\phi' = 0$ before the shock and the Rankine–Hugoniot conditions at the shock imply $\partial_x\phi' = 0$. Hence, Eq. (13) can be integrated after the shock only for $x \in (\Sigma, L)$ with the boundary conditions

$$\partial_x\phi' = 0 \text{ at the shock } \sigma, \quad \phi'(L) = 0 \text{ or } \partial_x\phi'(L) = 0.$$

### *Numerical Simulation*

A simple numerical test was made. A geometry is chosen: $S(x) = 1 + a(b) * \sin(\pi(x/L),\ x \in (0, L)$ with $a(0) = 0.2$ and $L = 1$. Stability is studied around the stationary state, i.e. $u(x)$ satisfying $ue^{-(1/2)u^2} = 0.35/S(x)$. Equation (13) is integrated up to $T = 10$ by an explicit scheme with all "primes" being derivatives with respect to $b$ with $a'$ such that $0.35(\partial_x S/S)' = \sin(2\pi(x/L))$.

Figure 1 shows $t \to \phi'(0, t)$, which is proportional to the displacement of the shock, and $x \to \phi'(x, T)$. This is a case where the displacement of the shock can be computed quite easily.

**THE TWO-DIMENSIONAL CASE**

Assume now that $\phi$, solution of Eq. (4) is a function of a scalar parameter $a$ via the data of the partial differential equation and that it has a shock $\Sigma(a)$. We wish to differentiate $\phi$ with respect to $a$. For clarity we will seek the result at $a = 0$.

Denote by $\alpha(x)a$ the distance in the direction $n_\Sigma$, normal to the shock $\Sigma$ pointing inside $\Omega_+$, between $\Sigma(a)$ and $\Sigma := \Sigma(0)$, i.e.

$$\Sigma(a) = \{x + a\alpha(x)n_\Sigma(x) : x \in \Sigma\}.$$

Denote by $\Omega_\pm$ the region before the shock and after the shock. Then, $I_D$ being the characteristic function of a set $D$,

$$u = u_- + (u_+ - u_-)I_{\Omega_+}$$

where $u_\pm$ are smooth functions. The derivative of $I_{\Omega_+}$ is a Dirac mass on $\Sigma$ (see Bardos and Pironneau, 2002). So if $\phi'$, $u'$ denotes the derivative of $\phi$, $u$ with respect to $a$, we have

$$u' = u'_- + (u'_+ - u'_-)I_{\Omega_+} - [u]\alpha\delta_\Sigma$$

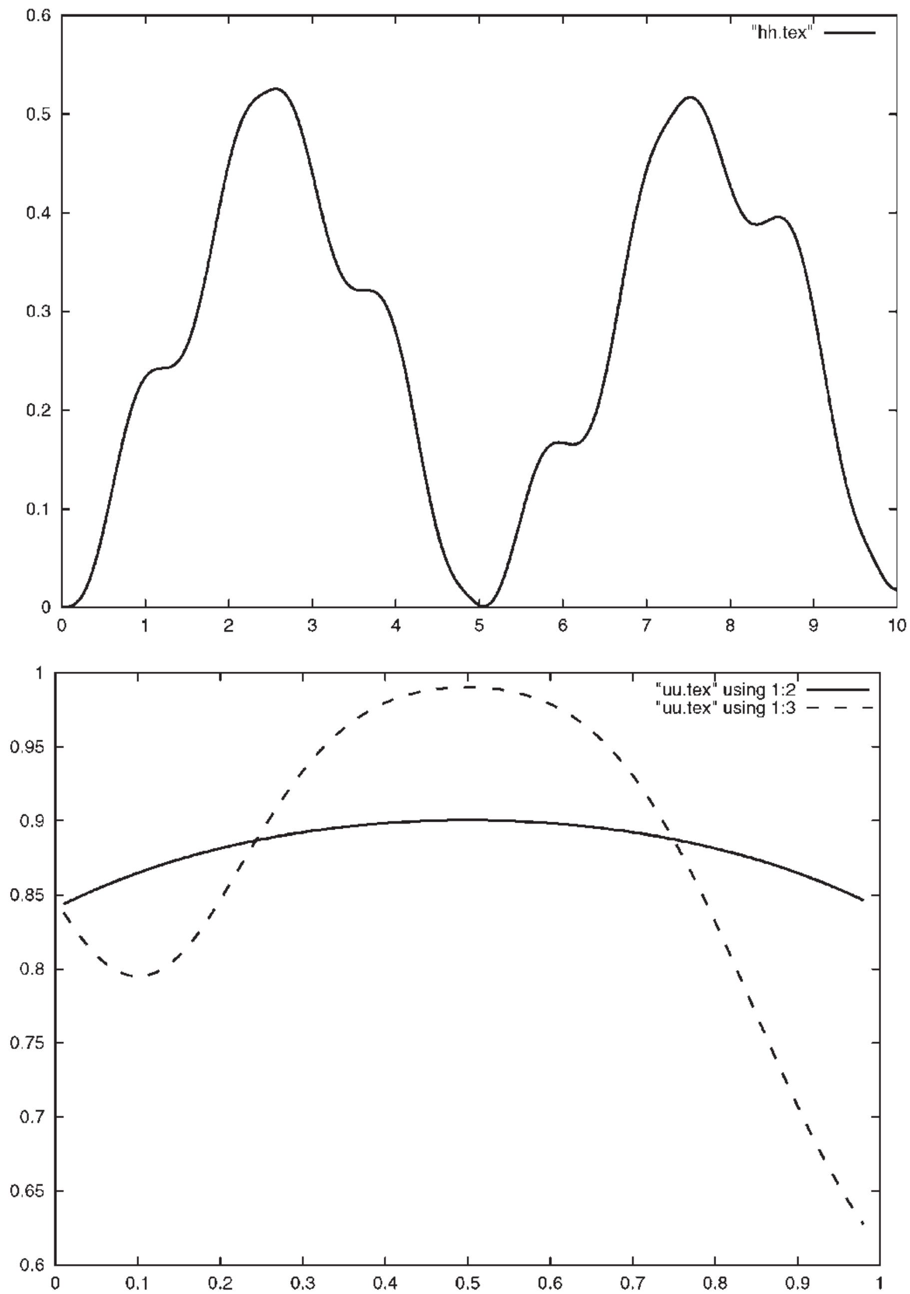


FIGURE 1 Top: Variation of the shock position (downstream to its initial value) versus time. Bottom: Water level before (smoother curve) and after the change in geometry, at final time.

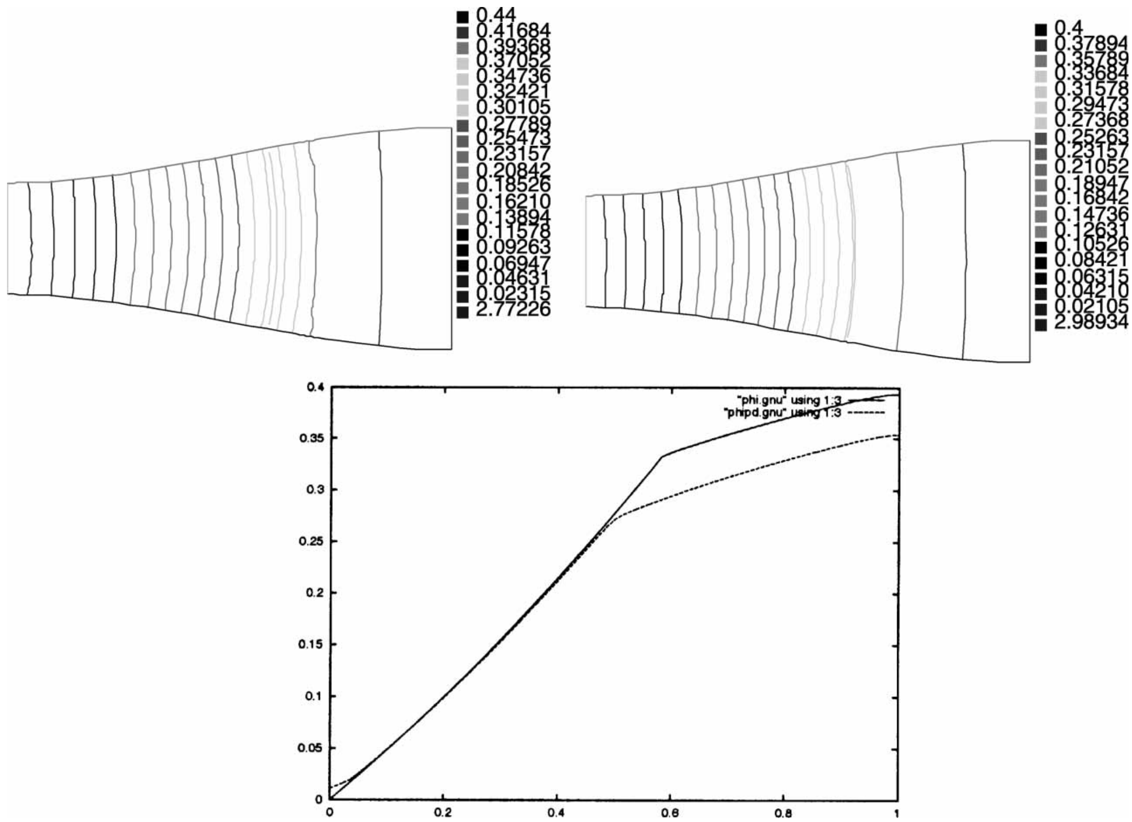


FIGURE 2 Level lines of $\phi$ with a changing potential at the outflow boundary going from 0.4 (left) to 0.44 (right). Bottom: Plot of $x \to \phi(x)|_{\Gamma_w}$.

and therefore if $u'$ has a Dirac mass on $\Sigma$, $\phi'$ must be *discontinuous* across $\Sigma$, and

$$[\phi']_\Sigma = -[u]_\Sigma \cdot n_\Sigma \alpha. \tag{14}$$

Away from the shock the transonic equation can be differentiated, giving:

$$\nabla\cdot(\rho'\nabla\phi\cdot\nabla w + \rho\nabla\phi') \equiv \nabla\cdot\left(\rho\left(1 - \frac{2\beta u\otimes u}{1-|u|^2}\right)\nabla\phi'\right) = 0. \tag{15}$$

But Eq. (15) makes no sense at the shock because a Dirac mass is multiplied by a discontinuous function.

In Bardos and Pironneau (2002) a formalism has been introduced to differentiate such functions. Suppose that $u$ and $v$ are discontinuous at $x(a)$. Then the following is expected:

$$u' = u'_a - [u]x'_s\delta(x-x_s) \quad v' = v'_a - [v]x'_s\delta(x-x_s)$$

$$(uv)' = (uv)'_a - [uv]x'_s\delta(x-x_s).$$

The trick is to define

$$\bar u = \left|\begin{array}{ll} \frac{1}{2}(u^+ + u^-) & \text{at the shock} \\ u & \text{elsewhere} \end{array}\right.$$

and then notice that "$(uv)' = \overline{\{u\}}v' + u'\overline{\{v\}}\bar v$" contains the identity "$[uv] = \{\bar u\}[v] + [u]\{\bar v\}$".

When $\beta = 5/2$, $\rho^2 = (1-u^2)^5 = AB$ with $A = (1-u^2)$, $B = C^2$, $C = A^2$,

$$\begin{aligned} 2\bar\rho\rho' &= A'\bar B + \bar A B' = -2\bar u u'\bar B + 2\bar A\bar C C' \\ &= -2\bar u u'\overline{(1-u^2)^4} + 4\,\overline{(1-u^2)}\;\overline{(1-u^2)^2}\,\bar A A' \\ &= -2\bar u u'\,(\overline{(1-u^2)^4} + 4\,\overline{((1-u^2))^2(1-u^2)^2}). \end{aligned} \tag{16}$$

Therefore, the linearized transonic equation is

$$\nabla\cdot(\bar M\nabla\phi') = 0 \text{ with } \bar M = (1-u^2)^{2.5}\left(1 - 2\beta\frac{u\otimes u}{1-u^2}\right) \text{ in } \Omega\backslash\Sigma \tag{17}$$

with $u = |\nabla\phi|$ and, at the shock $\Sigma$

$$\bar M = \overline{(1-u^2)^{2.5}} \times\left(I - \frac{\overline{(1-u^2)^4} + 4\,\overline{(1-u^2)}^2\,\overline{(1-u^2)^2}}{\overline{(1-u^2)^{2.5}}^2}\bar u\otimes\bar u\right). \tag{18}$$

Notice that both formulae for $\bar M$ agree in $\Omega\backslash\Sigma$.

In variational form, Eq. (4) is

$$\forall w \in H^1(\Omega) \int_\Omega \rho \nabla \phi \cdot \nabla w = \int_\Gamma g w.$$

When $\Omega = \Omega(a)$, assuming that $g = 0$ on the parts of $\Gamma$ which depend on $a$, the derivative of this formulation is (see Pironneau, 1983)

$$\forall w \in H^1(\Omega) \int_\Omega \bar{M} \nabla \phi' \cdot \nabla w = - \int_\Gamma (\rho \nabla \phi \cdot \nabla w) \alpha$$

which is also Proposition 1.

PROPOSITION 1 The differentiated transonic equation with respect to boundary variations $\alpha$ is a system for $(\phi'_S, [\phi'], x'_s)$, its regular part, its jump across the shock $\Sigma$ and the variation of the shock position:

$$\forall w \in H^1(\Omega) \int_{\Omega \backslash \Sigma} \bar{M} \nabla \phi' \cdot \nabla w - \int_\Sigma \bar{M} [\nabla \phi] \cdot n_\Sigma \frac{\partial w}{\partial n} x'_s = - \int_\Gamma (\rho \nabla \phi \cdot \nabla w) \alpha. \quad (19)$$

*Remark 2* (19) contains the Rankine–Hugoniot condition $[M \nabla \phi'] \cdot n_\Sigma = 0$. It is interpreted as

$$\nabla \cdot (M \nabla \phi') = 0, \quad [\bar{M} \nabla \phi'] \cdot n|_\Sigma = 0,$$
$$\bar{M} \nabla \phi' \cdot n|_\Gamma = -\rho \frac{\partial \phi}{\partial s} \frac{\partial \alpha}{\partial s}, \quad [\phi'] = -[u] \cdot n_\Sigma x'_S. \quad (20)$$

Comparison with the one-dimensional case shows that an additional integral condition is needed, like the total mass of fluid specified.

## NUMERICAL SIMULATION

Consider first a very simple case, that of a transonic divergent nozzle with supersonic inflow conditions and a difference of potential so that there is a shock in the flow. This example has been investigated in Bardos and Pironneau (2003) for sensitivity with respect to changes in the inflow or outflow boundary condition. We recall the results obtained. Upstream of the shock the PDE is hyperbolic and so the perturbation $\phi'$ is zero because it is entirely defined by the inflow conditions, which are zero, integrated on the streamlines.

Suppose we want to move the shock by a known quantity $x'_s$, then the value of $\phi'$ is known on $\Sigma$ and we must find the boundary condition $f$ which gives $\frac{\partial \phi'}{\partial n} = 0$ on $\Sigma$, for instance by solving

$$\min_f \left\{ \int_\Sigma (\phi' - \phi'_\Sigma)^2 : \nabla \cdot (M \nabla \phi') = 0, \text{ in } \Omega^+ \quad \frac{\partial \phi'}{\partial n}|_\Sigma = 0, \quad \frac{\partial \phi'}{\partial n}|_\Gamma = f \right\}.$$

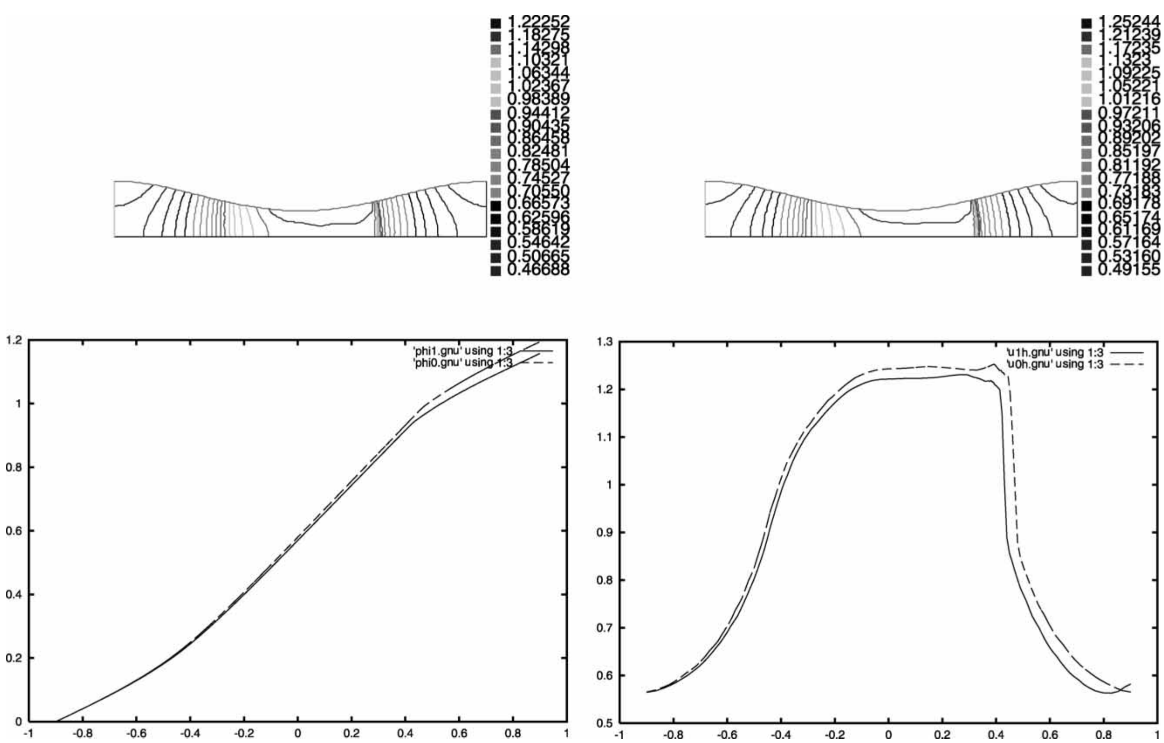


FIGURE 3 Mach lines with a changing Neumann outflow condition at the outflow. Bottom: Plot of $x \rightarrow \phi(x)|_{\Gamma_w}$ and $x \rightarrow (\partial \phi / \partial x)(x)|_{\Gamma_w}$ on the symmetry line $\Gamma_w$.

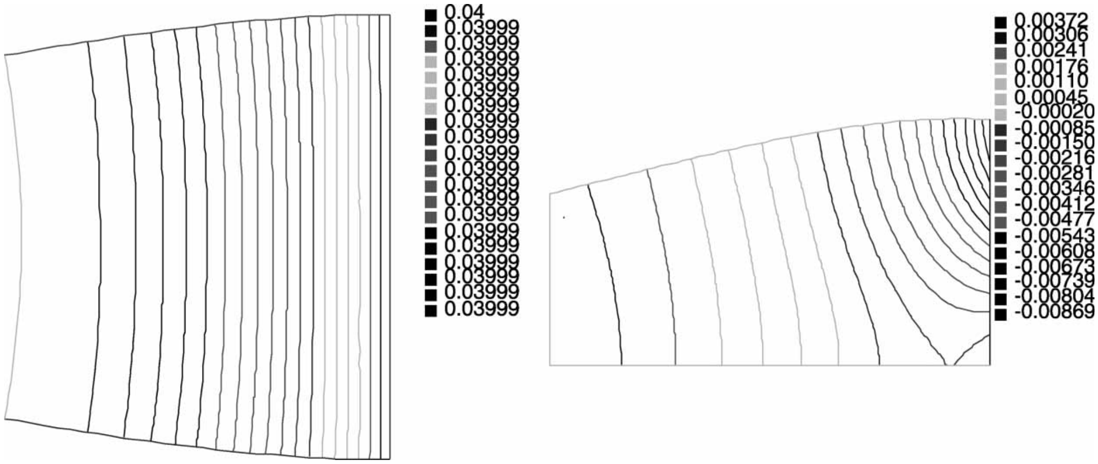


FIGURE 4 Level lines of $\phi'$ for the first and second test.

### Numerical Algorithm

As the transonic equation is nonlinear we used a fixed point algorithm with a small under-relaxation parameter (for instance 0.01); convergence is obtained with 50–200 iterations:

$$\nabla \cdot (\rho^m \nabla \phi^{m+1}) = 0.$$

To compute $\rho^{m+1}$ we compute the two roots $u_\pm$ of

$$(1-u^2)^\beta u = \left(1 - |\nabla \phi^{m+1}|^2\right)^\beta |\nabla \phi^{m+1}|$$

and set $\rho^{m+1} = (1-u_-^2)^\beta$ if $u$ decreases or if both $u$ and $\rho u$ grow on the streamline and set $\rho^{m+1} = (1-u_+^2)^\beta$ otherwise.

### The Divergent Nozzle

Using `freefem+`(http://www.freefem.org) we computed the solution of the transonic equation in a symmetric nozzle of equation

$$\Gamma_w = \left\{ (y(x), x) : y(x) = 1 + \frac{1}{8}(3x^2 - 2x^3), \quad x \in (0,1) \right\}.$$

We performed two computations with $u_i = 0.4$, and a potential difference of 0.4 or 0.44. The level curves of $\frac{\partial \phi}{\partial x}$ for these are reported on Fig. 3.2. They show that only the region after the shock changes, as predicted by the theory. Finally, we have solved numerically the PDE of $\phi'$ in the domain right to the shock with Neumann homogeneous conditions except on the outflow boundary where we have a Dirichlet condition equal to 0.04. The level lines are shown in Fig. 3.3. It predicts a shock displacement parallel and of distance $0.04/(0.6 - 0.2) = 0.1$ which is compatible with the experiments of Fig. 3.2.

### The Convergent Divergent Nozzle

The nozzle's equation is now

$$\Gamma_w = \{ (y(x), x) : y(x) = 1 + \frac{1}{8}(3x^2 - 2x^3),$$

$$x \in (-0.9, 0.9)\}.$$

The inflow and outflow boundary conditions are now

$$\rho u \cdot n = M * (1 - M^2)^\beta + \frac{k}{3}\left(\frac{1}{16} - y\right)$$

$$M = 0.4, \quad k = 0 \text{ or } 1.$$

Next we solved the equation for $\phi'$ with Neumann condition equal to zero at the shock boundary and equal to $\frac{1}{3}(\frac{1}{16} - y)$ at the outflow boundary. The level lines are shown in Figs. 3.3 and 4. This test is not very conclusive, it is at the border of numerical noise. The results on $\phi'$ predict a displacement of the shock at the symmetry line equal to 0.003/0.4, i.e. a little less than one per cent, which is about what we obtain numerically. However, there should not be any changes left of the shock and the numerical scheme does not show that.

### Perspective

The numerical strategy for nozzle flows may not work for wing profiles because the elliptic zone surrounds the hyperbolic zone and so there is no independent decomposition of the zones. Another problem with the method we have used here is that it is necessary to "track" the shock and remesh the domain with the shock as one boundary.

It would be convenient to put the linearized equation, the Rankine–Hugoniot conditions and the equation for the shock displacement into one single variational formulation. To do so we observed in Bernardi and Pironneau (2002) that

$$\int_\Omega \nabla\phi\cdot\nabla w = \int_{\partial D}\frac{\partial w}{\partial n_D} \Leftrightarrow \phi = I_D.$$

This leads us to try, for an appropriate $\gamma$

$$\int_\Omega M\nabla\phi'\nabla w = \int_\Sigma \gamma\frac{\partial w}{\partial n}$$

$$\forall w\in H=\{w\in H^1(\Omega):\ \ w|_{\Gamma_i\cup\Gamma_o}=0\}$$

$$\text{with}\ \ M=\rho\left(I-\frac{2\beta u\otimes u}{1-|u|^2}\right). \tag{21}$$

The solution would satisfy (15), and its Rankine–Hugoniot conditions, but it is hard to see that Eq. (14) would be satisfied because an integration by part in the neighborhood of $\Sigma$ gives a term which has no meaning, $M\delta_\Sigma$. However, in mixed form consider a similar problem with

$$U'\equiv M\nabla\phi' = \rho\left(I-\frac{2\beta u\otimes u}{1-|u|^2}\right)\nabla\phi',$$

that is (subject to non-homogeneous boundary conditions):

$$\int_\Omega (M^{-1}U'\cdot W+\phi'\nabla\cdot W)+\int_\Sigma \gamma W\cdot n = 0$$

$$\forall W\in H(\text{div},\Omega) \tag{22}$$

$$\int_\Omega \nabla\cdot U'w = 0\quad \forall w\in L^2(\Omega)$$

for which the unknowns are $U',\phi',\gamma$. Then $\phi'$ is also discontinuous and there is no ambiguity when integrated by part because $U'$ does not jump across $\Sigma$.

Two problems remain:

1. Justify Eq. (22) and show that it defines $\gamma$ indeed (if $M$ was elliptic everywhere $\gamma$ would be a data of the problem, but we have shown that it is not so for transonic flows, at least in the case of nozzles).
2. Solve numerically Eq. (22) with automatic numerical detection of $\Sigma$.

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