# A finite element problem issued from fictitious domain techniques

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**Abstract:** We propose a finite element discretization of a problem issued from the fictitious domain technique. We prove a priori error estimates for the discrete problem and present numerical experiments which are in good agreement with the analysis.

**Résumé:** Nous proposons une discrétisation par éléments finis d'un problème lié à la technique de domaine fictif. Nous prouvons des estimations d'erreur a priori pour le problème discret and présentons des expériences numériques qui confirment les résultats de l'analyse.

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## 1. Introduction.

The fictitious domain method (see for example [14][4][12][5][6][7]) is a technique which allows for solving elliptic problems in domains with unknown or moving boundaries without having to build a body fitted mesh. The main idea consists in embedding the computational domain into a larger domain with a smooth boundary and solving a modified problem obtained by extension of the data to this new domain. Most often, these data are extended by zero, so that when the physical boundary condition is not zero, the extended solution of the problem is discontinuous, but how does the finite difference and the finite element methods behave on discontinuous solutions; can one prove convergence?

Another motivation for this study is the analysis of front tracking by auxiliary functions in the simulation of multiphase flows for instance. There one must compute an indicator function which is constant in each phase [11] [12][13]. Therefore solving the problem below seems a key point for using this technique, since it contains the same type of discontinuity.

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitz–continuous boundary $\partial\Omega$. Let also $\mathcal{O}$ be an open set in $\mathbb{R}^d$ such that $\overline{\mathcal{O}}$ is contained in $\Omega$, with a boundary $\gamma$ which is either polygonal or polyhedral or of class $\mathcal{C}^{1,1}$. We consider the problem

$$\begin{cases} \Delta u = \Delta \chi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\chi$ stands for the characteristic function of $\mathcal{O}$. In this work, we prove the convergence of one of its finite element discretizations. An obvious solution of problem (1) is $u = \chi$, however we are interested in proving that it is the only one and in deriving error estimates which indicate that it can be computed accurately.

It can be observed that the solution of problem (1) cannot be smooth enough for problem (1) to admit a standard variational formulation. So we first write its formulation in the transposition sense of [15] and [17] and prove that it is well-posed. In contrast, as usual, the finite element discretization yields a discrete problem of variational type, but the energy norm of the discrete solution is not bounded independently of the discretization parameter. Nevertheless, optimal error estimates can be proven in a weak norm. From these estimates, it appears that the convergence is better when the grid is regular and not too fine in a neighbourhood of $\gamma$. Numerical experiments confirm the a priori estimates.

An outline of the paper is as follows.

- Section 2 is devoted to the analysis of the continuous problem.
- In Section 3, we describe the finite element problem and prove error estimates.
- Numerical experiments are presented in Section 4.

## 2. Analysis of the problem.

In what follows, we use the standard Hilbertian Sobolev spaces $H^s(\Omega)$ for any non-negative real number $s$. Similar spaces are defined on $\gamma$: for any $s > \frac{1}{2}$, $H^{s-\frac{1}{2}}(\gamma)$ is the space of traces on $\gamma$ of functions in $H^s(\Omega)$. We refer to [15, Chap. 1, Th. 11.7] for the definition of $H_{00}^{\frac{1}{2}}(\Gamma)$, for any part $\Gamma$ of $\partial\Omega$.

The first equation of problem (1) can be written in the distribution sense as

$$\forall v \in \mathcal{D}(\Omega), \quad \int_\Omega u(\Delta v)\, d\boldsymbol{x} = \int_\Omega \chi(\Delta v)\, d\boldsymbol{x} = \int_\mathcal{O} \Delta v\, d\boldsymbol{x}$$

where $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with compact support in $\Omega$.

Now, let $\boldsymbol{n}$ denote the unit normal to $\gamma$ pointing outside $\mathcal{O}$. So, integrating by parts leads to (note that this requires some regularity assumptions on $\gamma$, in order to give sense to the normal derivative $\partial_n v$ either globally on $\gamma$ if it is of class $\mathcal{C}^{1,1}$ or piecewise on each edge or face if $\mathcal{O}$ is a polygon or a polyhedron)

$$\forall v \in \mathcal{D}(\Omega), \quad \int_\Omega u(\Delta v)\, d\boldsymbol{x} = \int_\gamma \partial_n v\, d\tau.$$

This suggests a new formulation of problem (1), in the transposition sense of Lions, Magenes and Stampacchia (see [15, Chap. 2, §6] and [17] for instance): Find $u$ in $L^2(\Omega)$ such that

$$\forall v \in H^2(\Omega) \cap H_0^1(\Omega), \quad \int_\Omega u(\Delta v)\, d\boldsymbol{x} = \int_\gamma \partial_n v\, d\tau. \tag{2}$$

We first investigate the relation between problems (1) and (2).

**Proposition 1**. Any solution $u$ of problem (2) satisfies the first equation of (1) in the distribution sense and the boundary conditions of (1)
(i) in the sense of $H^{-\frac{1}{2}}(\partial\Omega)$ if $\partial\Omega$ is of class $\mathcal{C}^{1,1}$,
(ii) in the sense of the dual space of $H_{00}^{\frac{1}{2}}(\Gamma)$ for all edges $\Gamma$ of $\partial\Omega$ if $\Omega$ is a polygon, for all faces $\Gamma$ of $\partial\Omega$ if $\Omega$ is a polyhedron.

**Proof:** Letting $v$ run through $\mathcal{D}(\Omega)$ in (2) yields the first line of (1) in the distribution sense. Next, we introduce a neighbourhood $\mathcal{U}$ of $\partial\Omega$ in $\Omega$, such that the intersection of $\overline{\mathcal{U}}$ and $\overline{\mathcal{O}}$ is empty and that $\partial\mathcal{U} \setminus \partial\Omega$ is smooth enough. Note that $u$ belongs to $L^2(\Omega)$ and is harmonic in $\mathcal{U}$. So, since $\mathcal{D}(\overline{\mathcal{U}})$ is dense in the domain of the Laplace operator in $L^2(\mathcal{U})$ (see [8, §1.5.3] for instance), the trace of $u$ on $\partial\Omega$ can be defined by duality for all functions $v$ vanishing on $\partial\mathcal{U}$ with normal derivative vanishing on $\partial\mathcal{U} \setminus \partial\Omega$

$$\langle u, \partial_n v\rangle = \int_\mathcal{U} (u\Delta v - v\Delta u)\, d\boldsymbol{x}. \tag{3}$$

It can be checked that the mapping: $v \mapsto \partial_n v$, defined on the space $H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$, is onto $H^{\frac{1}{2}}(\partial\Omega)$ if $\partial\Omega$ is of class $\mathcal{C}^{1,1}$, or onto $H_{00}^{\frac{1}{2}}(\Gamma)$ for all edges or faces $\Gamma$ of $\partial\Omega$ if $\Omega$ is a polygon or a polyhedron, so that the trace of $u$ is defined in the sense stated in the proposition. Moreover, it follows from (2) and the fact that $u$ is harmonic in $\mathcal{U}$ that the right-hand side of (3) is zero, which yields the desired result.

**Remark:** When $\Omega$ is a polygon or a polyhedron, the trace of $u$ can be defined in a slightly stronger sense, as follows from [1]. Note that, in all cases, the trace of any solution $u$ of problem (1) in $L^2(\Omega)$ makes sense.

Next, we prove an *a priori* estimate for all solutions of this problem.

**Lemma 2**. If the domain $\Omega$ is convex, any solution $u$ of problem (2) satisfies

$$\|u\|_{L^2(\Omega)} \leq c\,\mathrm{meas}(\gamma)^{\frac{1}{2}}. \tag{4}$$

**Proof:** We have

$$\|u\|_{L^2(\Omega)} = \sup_{g\in L^2(\Omega)} \frac{\int_\Omega g u\,d\boldsymbol{x}}{\|g\|_{L^2(\Omega)}}.$$

As proven in [8, Thm 3.2.1.2], owing to the previous assumption on $\Omega$, for any $g$ in $L^2(\Omega)$, the solution $v$ in $H^1_0(\Omega)$ of the Poisson equation $\Delta v = g$ belongs to $H^2(\Omega)$ and satisfies

$$\|v\|_{H^2(\Omega)} \leq c\,\|g\|_{L^2(\Omega)}.$$

Using (2) and the trace theorem yields

$$\int_\Omega g u\,d\boldsymbol{x} = \int_\Omega u(\Delta v)\,d\boldsymbol{x} = \int_\gamma \partial_n v\,d\tau \leq c\,\|v\|_{H^2(\Omega)}\,\mathrm{meas}(\gamma)^{\frac{1}{2}}.$$

Combining all this leads to the desired estimate

We are now in a position to derive the well-posedness of problem (2).

**Proposition 3**. If the domain $\Omega$ is convex, problem (2) has a unique solution in $L^2(\Omega)$. Moreover, this solution satisfies (4).

**Proof:** First, we recall from [8, Thm 3.2.1.2] that, since $\Omega$ is convex, the Laplace operator $\mathcal{L}$ which associates with any function $g$ in $H^{-1}(\Omega)$ the solution $v$ in $H^1_0(\Omega)$ of $\Delta v = g$ is an isomorphism from $L^2(\Omega)$ onto $H^2(\Omega) \cap H^1_0(\Omega)$. So, the linear form:

$$g \;\mapsto\; \int_\gamma \partial_n(\mathcal{L}g)\,d\tau,$$

is continuous on $L^2(\Omega)$ and problem (2) is equivalent to: Find $u$ in $L^2(\Omega)$ such that

$$\forall g \in L^2(\Omega), \quad \int_\Omega u\,g\,d\boldsymbol{x} = \int_\gamma \partial_n(\mathcal{L}g)\,d\tau. \tag{5}$$

So, the existence and uniqueness of a solution of problem (2) follows from the Riesz theorem. Clearly, this solution satisfies (4).

**Remark:** A counter-example, due to Moussaoui [16], proves that the convexity assumption is necessary for proving uniqueness of the solution in $L^2(\Omega)$. However, when $\Omega$ is a non-convex polygon or polyhedron, it can be checked (see [8, Chap. 4] and [3]) that there exists a real number $s_\Omega$, $0 < s_\Omega < \frac{1}{2}$, only depending on the geometry of $\Omega$, such that the Laplace operator introduced in the previous proof is an isomorphism from $H^{-s_\Omega}(\Omega)$ onto $H^{2-s_\Omega}(\Omega)$ (in the case of a polygon, i.e. in dimension $d = 2$, this $s_\Omega$ is $> 1 - \frac{\pi}{\omega}$, where $\omega$ denotes the largest angle of $\Omega$). Thus, the modified problem: Find $u$ in $H^{s_\Omega}(\Omega)$ such that

$$\forall v \in H^{2-s_\Omega}(\Omega) \cap H_0^1(\Omega), \quad \langle u, \Delta v \rangle = \int_\gamma \partial_{\boldsymbol{n}} v \, d\tau, \tag{6}$$

has a unique solution in $H^{s_\Omega}(\Omega)$.

The conclusion of this section is that if the domain $\Omega$ is convex then problem (1) and (2) are equivalent when the solution of (1) is sought for in $L^2(\Omega)$. Indeed propositon 1 and 3 show that the unique solution of problem (2) also solves problem (1) and according to the remark above there is at most one solution in $L^2(\Omega)$. Of course this solution coincides with $\chi$.

## 3. Discretization by finite elements.

From now on, we assume that $\Omega$ is a polygon or a polyhedron. Let $(\mathcal{T}_h)_h$ denote a regular family of triangulations of $\Omega$ by (closed) triangles or tetrahedra, in the sense of [2, §17]. As usual, $h$ stands for the maximal diameter of the elements of $\mathcal{T}_h$. We also denote by $\mathcal{T}_h^\gamma$ the set of elements $K$ of $\mathcal{T}_h$ such that the intersection of $K$ and $\gamma$ has a positive measure in $\mathbb{R}^{d-1}$.

In what follows, $c$ stands for a generic constant that may vary from one line to the other but is always independent of $h$.

For each $h$ and for a fixed positive integer $k$, we consider the space

$$X_h = \big\{v_h \in H_0^1(\Omega);\ \forall K \in \mathcal{T}_h,\ v_{h\,|K} \in \mathcal{P}_k(K)\big\},$$

where $\mathcal{P}_k(K)$ denotes the space of restrictions to $K$ of polynomials with $d$ variables and total degree $\leq k$.

We propose the following discrete problem: Find $u_h$ in $X_h$ such that

$$\forall v_h \in X_h, \quad \int_\Omega \mathbf{grad}\, u_h \cdot \mathbf{grad}\, v_h \, d\boldsymbol{x} = -\int_\gamma \partial_n v_h \, d\tau. \tag{7}$$

To investigate the properties of this problem, we begin with a technical lemma.

**Lemma 4.** For any element $K$ in $\mathcal{T}_h^\gamma$ with diameter $h_K$ and for any $v_h$ in $X_h$, the following estimate holds

$$\Big| \int_{K\cap\gamma} \partial_n v_h \, d\tau \Big| \leq c \,\mathrm{meas}(K\cap\gamma)\, h_K^{-\frac{d}{2}}\, |v_h|_{H^1(K)}. \tag{8}$$

**Proof:** Let $\hat{K}$ be a reference triangle, $F_K$ one of the affine mappings that maps $\hat{K}$ onto $K$. We denote by $\hat{e}$ the image of $e = K \cap \gamma$ by $F_K^{-1}$. By going to the reference element $\hat{K}$ and with the usual notation $\hat{v}$ for any function $v \circ F_K$, we have

$$\Big| \int_{K\cap\gamma} \partial_n v_h \, d\tau \Big| \leq c \,\mathrm{meas}(e)\, h_K^{-1} \int_{\hat{e}} |\mathbf{grad}\, \hat{v}_h| \, d\hat{\tau}.$$

From the trace theorem it can be checked (see [5] for the result in the case of dimension $d = 2$, the proof easily extends to the case of dimension $d = 3$) that, for a constant $c^*$ independent of the geometry of $\hat{e}$,

$$\forall \varphi \in H^1(\hat{K}), \quad \int_{\hat{e}} |\varphi| \, d\hat{\tau} \leq c^* \, \|\varphi\|_{H^1(\hat{K})}. \tag{9}$$

By using the equivalence of norms on $\mathcal{P}_{k-1}(\hat{K})$, we also derive

$$\|\mathbf{grad}\, \hat{v}_h\|_{H^1(\hat{K})^d} \leq \hat{c}\, \|\mathbf{grad}\, \hat{v}_h\|_{L^2(\hat{K})^d} = \hat{c}\, |\hat{v}_h|_{H^1(\hat{K})}.$$

Combining all this and using the regularity of $(\mathcal{T}_h)_h$ yield

$$\Big| \int_{K\cap\gamma} \partial_n v_h \, d\tau \,\Big| \le c\,\mathrm{meas}(e)\, h_K^{-1}\, |\hat{v}_h|_{H^1(\hat{K})} \le c\,\mathrm{meas}(e)\, h_K^{-\frac{d}{2}}\, |v_h|_{H^1(K)},$$

which is the desired result.

The main properties of problem (7) are stated in the next proposition. In view of estimate (8), we set

$$\ell_h(\gamma) = \max_{K\in\mathcal{T}_h^\gamma} \mathrm{meas}(K\cap\gamma)^{\frac{1}{2}}\, h_K^{-\frac{d}{2}}. \tag{10}$$

It is readily checked that, except for some rather special geometries of $\mathcal{O}$ or at least when $h$ is small enough,

$$\ell_h(\gamma) \le c\,\big(\min_{K\in\mathcal{T}_h} h_K\big)^{-\frac{1}{2}}, \tag{11}$$

however, it can be much smaller than this bound.

**Proposition 5**. Problem (7) has a unique solution $u_h$ in $X_h$. Moreover this solution satisfies

$$\|u_h\|_{H^1(\Omega)} \le c\,\ell_h(\gamma)\,\mathrm{meas}(\gamma)^{\frac{1}{2}}. \tag{12}$$

**Proof:** Problem (7) results into a square linear system and, when the right-hand side is zero, the solution is zero. So it has a unique solution. To prove estimate (12), we take $v_h$ equal to $u_h$ in (7), which yields

$$|u_h|^2_{H^1(\Omega)} \le \sum_{K\in\mathcal{T}_h^\gamma} \Big| \int_{K\cap\gamma} \partial_n u_h\, d\tau \,\Big|.$$

Using Lemma 4 yields

$$|u_h|^2_{H^1(\Omega)} \le c \sum_{K\in\mathcal{T}_h^\gamma} \mathrm{meas}(K\cap\gamma)\, h_K^{-\frac{d}{2}}\, |u_h|_{H^1(K)} \le c\,\ell_h(\gamma) \sum_{K\in\mathcal{T}_h^\gamma} \mathrm{meas}(K\cap\gamma)^{\frac{1}{2}}\, |u_h|_{H^1(K)}.$$

We conclude thanks to Cauchy–Schwarz and Poincaré inequalities.

We are now interested in the error estimate.

**Theorem 6**. If the domain $\Omega$ is convex, the following error estimate holds between the unique solution $u$ of problem (2) and the solution $u_h$ of problem (7):

$$\|u-u_h\|_{L^2(\Omega)} \le c\,\ell_h(\gamma)\, h\,\mathrm{meas}(\gamma)^{\frac{1}{2}}. \tag{13}$$

**Proof:** As usual, we have

$$\|u-u_h\|_{L^2(\Omega)} = \sup_{g\in L^2(\Omega)} \frac{\int_\Omega (u-u_h) g\, d\boldsymbol{x}}{\|g\|_{L^2(\Omega)}}.$$

Next, since $\Omega$ is convex, using once more [8, Thm 3.2.1.2] yields that, for any $g$ in $L^2(\Omega)$, the solution $v$ in $H_0^1(\Omega)$ of the Poisson equation $\Delta v = g$ belongs to $H^2(\Omega)$ and satisfies

$$\|v\|_{H^2(\Omega)} \leq c\, \|g\|_{L^2(\Omega)}.$$

Next, we have

$$\int_\Omega (u - u_h) g \, d\boldsymbol{x} = \int_\Omega u(\Delta v) \, d\boldsymbol{x} + \int_\Omega \mathbf{grad}\, u_h \cdot \mathbf{grad}\, v \, d\boldsymbol{x},$$

whence, for any $v_h$ in $X_h$,

$$\int_\Omega (u - u_h) g \, d\boldsymbol{x} = \int_\gamma \partial_n (v - v_h) \, d\tau + \int_\Omega \mathbf{grad}\, u_h \cdot \mathbf{grad}\, (v - v_h) \, d\boldsymbol{x}.$$

Thanks to Proposition 5, the last term is bounded by

$$\int_\Omega \mathbf{grad}\, u_h \cdot \mathbf{grad}\, (v - v_h) \, d\boldsymbol{x} \leq c\, \ell_h(\gamma) \operatorname{meas}(\gamma)^{\frac{1}{2}} \, |v - v_h|_{H^1(\Omega)},$$

so that taking $v_h$ equal to the image of $v$ by the Lagrange interpolation operator yields

$$\int_\Omega \mathbf{grad}\, u_h \cdot \mathbf{grad}\, (v - v_h) \, d\boldsymbol{x} \leq c\, \ell_h(\gamma)\, h \operatorname{meas}(\gamma)^{\frac{1}{2}} \, |v|_{H^2(\Omega)}.$$

In order to bound the other term, with the same notation as in the proof of Lemma 4, we have for all $K$ in $\mathcal{T}_h^\gamma$,

$$\int_{K \cap \gamma} \partial_n (v - v_h) \, d\tau \leq c \operatorname{meas}(e)\, h_K^{-1} \int_{\hat{e}} |\mathbf{grad}\, (\hat{v} - \hat{v}_h)| \, d\hat{\tau}.$$

Thanks to (9), this yields

$$\int_{K \cap \gamma} \partial_n (v - v_h) \, d\tau \leq c \operatorname{meas}(e)\, h_K^{-1} \, \|\hat{v} - \hat{v}_h\|_{H^2(\hat{K})}.$$

Noting that the Lagrange interpolation operator on $\hat{K}$ is continuous on $H^2(\hat{K})$ and preserves the polynomials in $\mathcal{P}_1(\hat{K})$, we obtain

$$\int_{K \cap \gamma} \partial_n (v - v_h) \, d\tau \leq c \operatorname{meas}(e)\, h_K^{-1} \, |\hat{v}|_{H^2(\hat{K})}.$$

So going back to $K$ gives

$$\int_{K \cap \gamma} \partial_n (v - v_h) \, d\tau \leq c \operatorname{meas}(e)\, h_K^{1-\frac{d}{2}} \, |v|_{H^2(K)}.$$

As previously, summing over all $K$ in $\mathcal{T}_h^\gamma$ leads to

$$\int_\gamma \partial_n (v - v_h) \, d\tau \leq c\, \ell_h(\gamma)\, h \operatorname{meas}(\gamma)^{\frac{1}{2}} \, |v|_{H^2(\Omega)}.$$

This concludes the proof.

So, when the family $(\mathcal{T}_h)_h$ is quasi-uniform (as defined in [2, §17]), the error is smaller than $c\, h^{\frac{1}{2}}$. Otherwise, the convergence is obtained with the non restrictive condition

$$\lim_{h \to 0} \, (\min_{K \in \mathcal{T}_h} h_K)^{-1} \, h^2 = 0.$$

Moreover, this convergence can be extended to the case of a non-convex polygon $\Omega$, when the famity $(\mathcal{T}_h)_h$ is quasi-uniform and the solution $u$ of problem (2) belongs to $H^s(\Omega)$, $s > 1 - \frac{\pi}{\omega}$. However, for the fictitious domain technique, the choice of $\Omega$ is most often a matter of convenience, so that the convexity assumption is not restrictive.

## 4. Numerical experiments.

Problem (7) is solved numerically in the two-dimensional case with polynomials of degree one on triangles. The domain $\Omega$ is the unit square $]-1,1[^2$, and $\mathcal{O}$ is the disk with center $(0,0)$ and radius 0.5 (here approximated by a regular polygon with lengths of edges of order $h$). A sequence of quasi-uniform meshes is generated by freefem+ [10], and the solution is also computed with freefem+.

Next, the same computation is performed again but with a mesh adaptation step via the metric defined by the numerical solution of the equation (see Hecht [9] for more details on the adaptivity criteria). Note that, here again, the circle is never part of the mesh. These meshes are non isotropic and are finer in the normal direction to the circle. Then the error is improved greatly, and, even if this is not forecast by the estimate of Theorem 1, it is not in contradiction with it.

In Figure 1, the solution is presented for the two types of meshes, the curves of isovalues of the error are given in Figure 2. The meshes are drawn in Figure 3. Finally the curves of the error in the $L^2(\Omega)$ norm are described in Figure 4 and it can be checked that the error behaves very precisely as $h^{\frac{1}{2}}$.

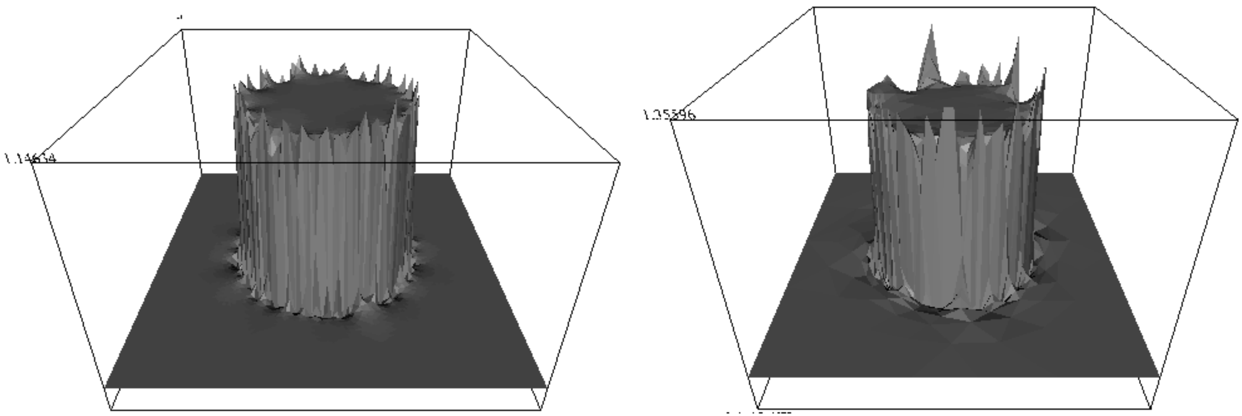

Figure 1: Computation without (left) and with (right) mesh adaptation: the solution is displayed for the 6th quasi-uniform mesh (5000 vertices) and for the 7th adapted mesh (900 vertices).

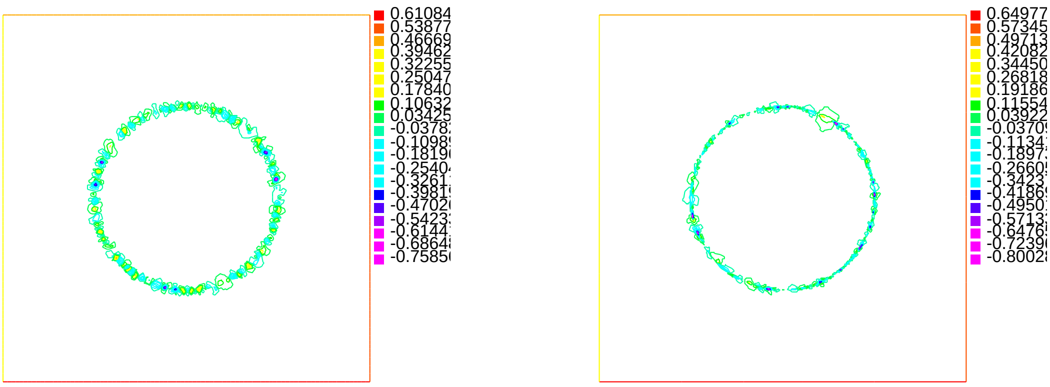

Figure 2: Error isolines on the left for the 10th quasi-uniform mesh (14500 vertices) and on the right for the 8th adapted mesh (1200 vertices).

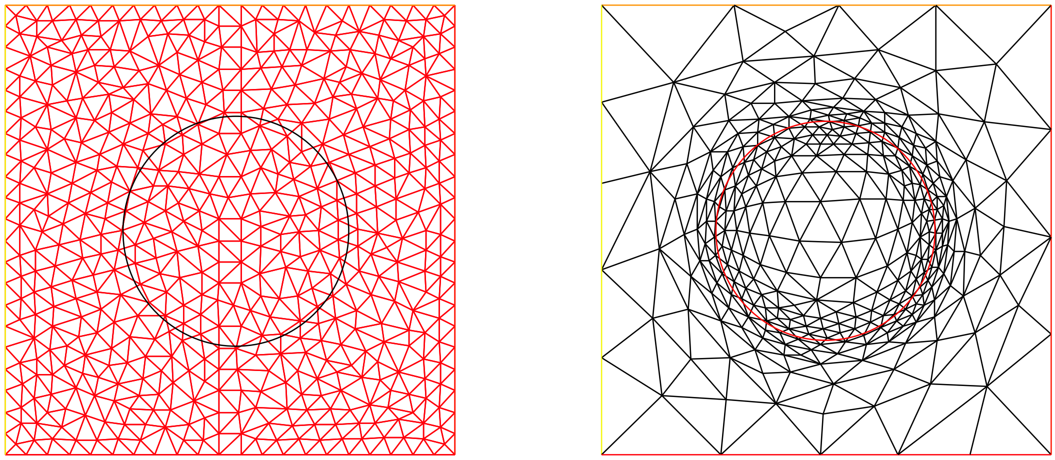

Figure 3: Adapted mesh after 3 iterations (right) compared with a standard unstructured mesh (left).

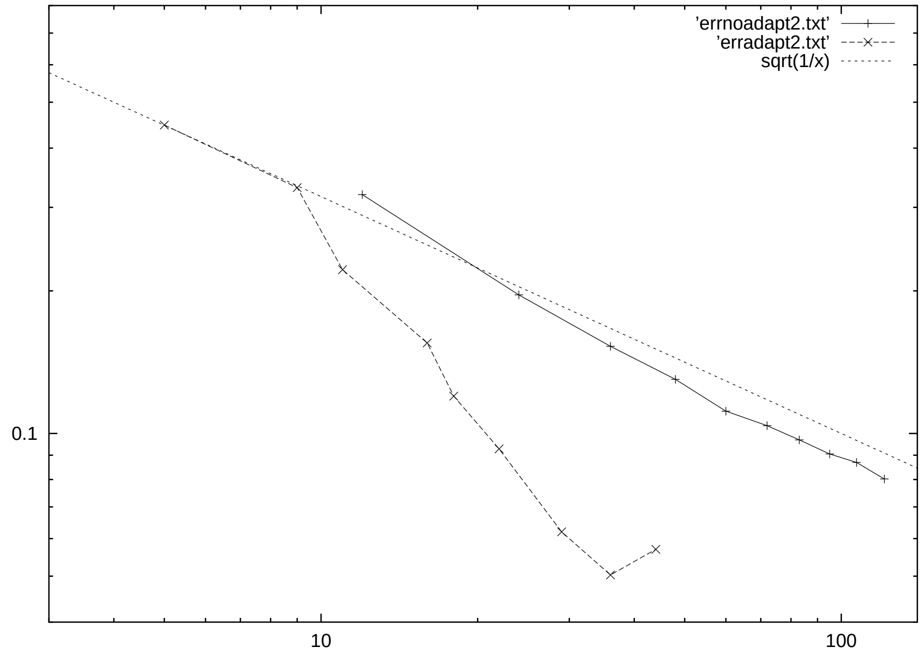


Figure 4: Error in $L^2(\Omega)$ plot in log-log scale with (dashed curve) and without (plain curve) mesh adaptation and comparison with the dotted line $y = 1/\sqrt{x}$.

## References


[1] C. Bernardi, M. Dauge, Y. Maday — Compatibilité de traces aux arêtes et coins d'un polyèdre, *C.R. Acad. Sc. Paris* **331**, série I (2000), 679–684.

[2] P.G. Ciarlet — *Basic Error Estimates for Elliptic Problems*, in the *Handbook of Numerical Analysis*, Vol. II, P.G. Ciarlet & J.-L. Lions eds., North-Holland (1991), 17–351.

[3] M. Dauge — Neumann and mixed problems on curvilinear polyhedra, *Integr. Equat. Oper. Th.* **15** (1992), 227–261.

[4] H. Fujita, H. Kawahara, H. Kawarada — Distribution theoretic approach to fictitious domain method for Neumann problems, *East-West J. Numer. Math.* **3** No.2 (1995), 111–126.

[5] V. Girault, R. Glowinski — Error analysis of a fictitious domain method applied to a Dirichlet problem, *Japan J. Indust. Appl. Math.* **12** (1995), 487–514.

[6] R. Glowinski, T. Pan, J. Périaux — A fictitious domain method for Dirichlet problems and applications. *Comput. Meth. Appl. Mech. Engrg.* **111** (1994), 283–303.

[7] R. Glowinski, T. Pan, J. Périaux — A fictitious domain method for external incompressible viscous flow modelled by Navier-Stokes equations. *Comput. Meth. Appl. Mech. Engrg.* **111** (1994), 133–148.

[8] P. Grisvard — *Elliptic Problems in Nonsmooth Domains*, Pitman (1985).

[9] F. Hecht — Bidimensional adapted mesh generator, I.N.R.I.A. Report, Rocquencourt (1999).

[10] F. Hecht, O. Pironneau — Multiple meshes and the implementation of freefem+, I.N.R.I.A. Report, Rocquencourt (1999).

[11] H. Kawarada, H. Fujita, H. Suito — Wave breaking phenomena on the shore and coastal ecosystems, *GAKUTO International Series, Mathematical Sciences and Applications* **12** (1999), 65–74.

[12] H. Kawarada, H. Suito — Numerical method for a free surface flow on the basis of the fictitious domain method, *East-West J. Numer. Math.* **5** No. 1 (1997), 57–66.

[13] H. Kawarada, H. Suito — Multi-phase flow with reaction, to appear in the *Proceedings of France-Japan Joint Workshop on Computational Methods for Control Applications*, Tokyo, Japan (1998).

[14] Yu.A. Kuznetsov, G.I. Marchuk — *Méthodes itératives et fonctionnelles quadratiques*, in *Méthodes Mathématiques de l'Informatique 4: sur les méthodes numériques en sciences physiques et économiques*, J.-L. Lions & G.I. Marchuk eds., Dunod (1974), 3–132.

[15] J.-L. Lions, E. Magenes — *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod (1968).

[16] M.A. Moussaoui — Sur l'approximation des solutions du problème de Dirichlet dans un ouvert avec coins, in *Singularities and Constructive Methods for their Treatment*, P. Grisvard, W. Wendland & J.R. Whiteman eds., Lecture Notes in Mathematics **1121**, Springer-Verlag (1984), 199–206.

[17] G. Stampacchia — *Équations elliptiques du second ordre à coefficients discontinus*, Presses de l'Université de Montréal (1965).