ANALYSIS OF A CHIMERA METHOD

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ABSTRACT. Chimera is a variant of Schwarz' algorithm which is used in CFD to avoid meshing complicated objects. In a previous publication [4] we proposed an implementation for which convergence could be shown except that ellipticity was not proved for the discretized bilinear form with quadrature rules. Here we prove that the bilinear form of the discrete problem is strongly elliptic without compatibility condition for the mesh of the subdomains in their region of intersection.

Résumé

Chimera est une variante de l'algorithme de Schwarz utilisé en mécanique des fluides numérique afin d'éviter le maillage d'objet trop compliqué. Dans [4] nous avions proposé une implémentation dont on peut démontrer la convergence sauf pour l'ellipticité que nous avions laissé en conjecture. On la démontre ici, sans hypothèse de compatibilité entre les maillages des différentes regions.

1. VERSION FRANÇAISE ABRÉGÉE

La méthode Chimère[5] vise à résoudre des équations aux dérivées partielles dans Ω par décomposition en sous-domaines $\{\Omega_i\}_1^N$ avec recouvrement afin d'eviter d'avoir à utiliser un maillage global. L'algorithme proposé par Steger est en fait un algorithme de Schwarz. Dans [4] nous avons proposé de décomposer la solution u de l'EDP en N parties chacunes dans $H_0^1(\Omega_i)$. Nous avons montrer que la méthode avec régularisation converge. Sa discretization (5) pose un problème numérique d'implémentation car on doit calculer une intégrale d'un produit de fonctions définies sur deux triangulations différentes.
Nous montrons ici que si les sommets des triangles des deux triangulations sont les points de quadrature alors la méthode converge (avec ordre optimal) car le lemme de Strang s'applique [4] et la forme bilineaire discrète est fortement elliptique. Ce point technique et difficile fait l'objet de cette note. La formule d'intégration est définie en (10) et le résultat démontré est en (12). On montre aussi (cf Lemme 1) que la décomposition d'une fonction w en une somme de fonctions affines par morceaux sur chacun des maillages est en quelque sorte unique.
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2. INTRODUCTION

The Chimera method [5] was proposed to bypass the difficulty of generating general unstructured meshes for complex objects like airplanes. It is also quite convenient to improve accuracy of the fictitious domain method as it provides a corrector solved locally on a body-fitted fine mesh around each complex object independently. The method is presented in dimension two on the Laplace equation, but it applies to any elliptic system and also in 3d.

More precisely let u_e be the (exact) solution of

$$-\Delta u_e = f \text{ in } \Omega, \quad u_e = 0 \text{ on } \Gamma \qquad (\Gamma \equiv \partial\Omega) \tag{1}$$

or in variational form, with self explanatory notations

$$(\nabla u_e, \nabla \hat{u})_\Omega = (f, \hat{u})_\Omega \quad \forall \hat{u} \in H_0^1(\Omega) \tag{2}$$

Assume that we are given a decomposition of $\Omega = \Omega_1 \cup \Omega_2$ such that $\Omega_1 \cap \Omega_2$ is not empty.

Like the Schwarz algorithm, Chimera computes u_e as the limit of $\{u_e^n, v_e^n\}$ given by

$$\left| \begin{array}{l} (\nabla u_e^{n+1}, \nabla \hat{u})_{\Omega_1} = (f, \hat{u})_{\Omega_1} \; \forall \hat{u} \in H_0^1(\Omega_1) \quad u_e^{n+1} - \tilde{v}_e^n \in H_0^1(\Omega_1) \\ (\nabla v_e^{n+1}, \nabla \hat{v})_{\Omega_2} = (f, \hat{v})_{\Omega_2} \; \forall \hat{v} \in H_0^1(\Omega_2) \quad v_e^{n+1} - \tilde{u}_e^n \in H_0^1(\Omega_2) \end{array} \right. \tag{3}$$

where the tilda denotes any extension in $H_0^1(\Omega)$.

Let $\mathcal{T}_h$ be a triangulation of Ω_1 and $\mathcal{K}_H$ a triangulation of Ω_2. Let V_h and V_H be the corresponding spaces of piecewise linear continuous functions. We shall denote by V_{0h} and V_{0H} the corresponding subspaces of $H_0^1(\Omega_1)$ and $H_0^1(\Omega_2)$, respectively.

A realistic way of writing the discrete analogue of (3) in the finite element subspaces is to proceed *by translation*: we first introduce suitable numerical integration formulae $(\;,\;)_h$ and $(\;,\;)_H$ in Ω_1 and Ω_2 respectively, and then, at each step, we solve the problem: find $\{u_0^{n+1}, v_0^{n+1}\} \in V_{0h} \times V_{0H}$ solution of

$$\left| \begin{array}{l} (\nabla(u_0^{n+1} + v_e^n), \nabla \hat{u})_h = (f, \hat{u})_h \quad \forall \hat{u} \in V_{0h} \\ (\nabla(v_0^{n+1} + u_e^n), \nabla \hat{v})_H = (f, \hat{v})_H \quad \forall \hat{v} \in V_{0H} \end{array} \right. \tag{4}$$

and we set $\{u_e^{n+1}, v_e^{n+1}\} = \{u_0^{n+1} + v_e^n, v_0^{n+1} + u_e^n\}$. In (5) (and in all the sequel) we identify a function in $H_0^1(\Omega_i)$ $(i = 1, 2)$ with its extension by zero to the whole Ω.

Notice that the recurrence relation between $\{u_e^{n+1}, v_e^{n+1}\}$ and $\{u_e^n, v_e^n\}$ shows that both are sums of one function in V_{0h} plus one function in V_{0H} so instead of (4) we shall consider:

Problem: Find $\{u^{n+1}, v^{n+1}\} \in V_{0h} \times V_{0H}$ solution of

$$\left| \begin{array}{l} (\nabla(u^{n+1} + v^n), \nabla \hat{u})_h = (f, \hat{u})_h \quad \forall \hat{u} \in V_{0h} \\ (\nabla(v^{n+1} + u^n), \nabla \hat{v})_H = (f, \hat{v})_H \quad \forall \hat{v} \in V_{0H} \end{array} \right. \tag{5}$$

In [4], [3] it is shown that the method converges if both equations are regularized by adding terms like $\beta(u^{n+1}-u^n,\hat{u})$ and $\beta(v^{n+1}-v^n,\hat{v})$ respectively to the first and second equation in (5); we have also proposed to use Gauss quadratures on the gradients, but a proof of convergence in the general case was not given. Here we take up the idea but put the quadrature points at the vertices instead of inside the triangles and show that the method works in a rather general setting; we point out however that the previous integration formula allowed an alternative implementation (by penalty, i.e. putting a large number on the diagonal of the discrete linear system (see [3])) that is not allowed here.

In the following Section, we describe in more details the assumptions on the decompositions $\mathcal{T}_h$ and $\mathcal{K}_H$, and the numerical integration formula. Then, in the final Section, we prove a basic ellipticity result for the corresponding bilinear form, and we indicate how this implies the convergence of the iterative methods.

3. Assumptions on the decompositions

Let, as before, $\mathcal{T}_h$ be a triangulation of Ω_1 and $\mathcal{K}_H$ a triangulation of Ω_2. We assume that both decompositions are regular and quasi-uniform, in the sense that, if h_M and h_m are the maximum and minimum edges in $\mathcal{T}_h$ and H_M and H_m are the maximum and minimum edges in $\mathcal{K}_H$, then there exists two constants C_T and C_K such that

$$h_M \leq C_T h_m \qquad H_M \leq C_K H_m. \tag{6}$$

Without loss of generality we can also assume, to fix the ideas, that

$$h_M \leq H_M. \tag{7}$$

Remark 1. *To be precise, we should speak of two sequences of decompositions in the above assumptions. The meaning of (6) would then be that the constants C_T and C_K do not depend on the choice of the decomposition in the sequence.*

In what follows T will denote a generic triangle of the triangulation $\mathcal{T}_h$ of Ω_1 and K a generic triangle of the triangulation $\mathcal{K}_H$ of Ω_2.

Let $q_T^1,...,q_T^R$ be the vertices of $\mathcal{T}_h$, and $q_K^1,...,q_K^S$ be the vertices of $\mathcal{K}_H$. It will be convenient also to denote the same by $q^i(T)$ (resp $q^i(K)$), $i=1,2,3$ when we refer to the 3 vertices of a triangle.

A crucial assumption that we make is that each node q_T of $\mathcal{T}_h$ is internal to a triangle K and each node q_K of $\mathcal{K}_H$ is internal to a triangle T. This, at first sight, sounds rather restrictive. However, it is clear that one can always reach such a situation by a very small change in the position of the vertices. As we shall see in the next section, a vertex that is *very close* to an edge of the other decomposition will *not* affect the overall quality of the method; in fact this assumption is necessary only for notational convenience as it makes our quadrature definition unique.

The following lemma will be useful in the sequel.

Lemma 1. *If two functions $u \in V_h$ and $v \in V_H$ coincide on a subset $\mathcal{S}$ of $\Omega_1 \cap \Omega_2$, then both u and v are linear (not just piecewise linear) in $\mathcal{S}$.*

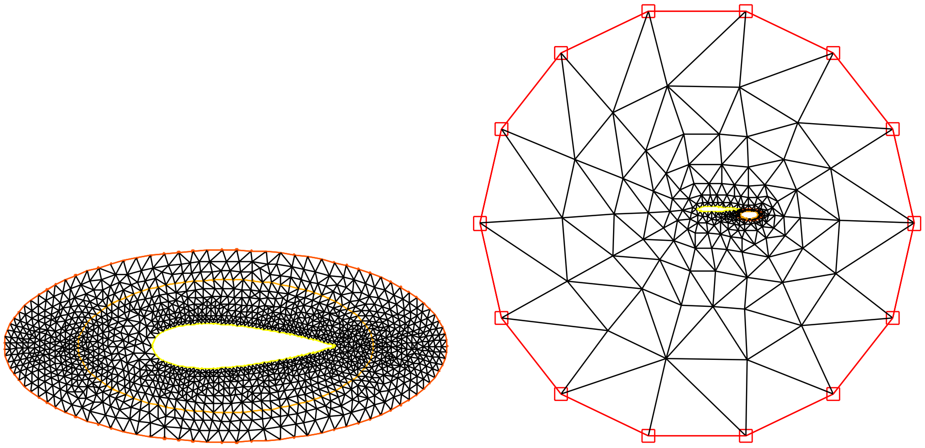

FIGURE 1. *To compute the stream function around a two-pieces airfoil, namely solution of $\Delta\psi = 0$ with Dirichlet data by the Chimera method (i.e. Schwarz' algorithm) We build a finer mesh around the smaller airfoil (on the left) and a coarse mesh for the rest of the domain with an elliptic hole in place of the small airfoil (the scale for both domains is not the same on this picture). The whole domain is the union of the fine and coarse domains.*

Proof As $\Delta u = \Delta v$ it is a distribution with support on the edges of $\mathcal{T}_h$ and a distribution with support on the edges of $\mathcal{K}_H$. But the two sets have in common only isolated points, where an edge of $\mathcal{T}_h$ crosses an edge of $\mathcal{K}_H$

We finally observe that Δu is in $H^{-1}(\Omega)$ (actually, in $H^s(\Omega)$ for $s < -1/2$), and hence, as a distribution, its support cannot contain isolated points. Consequently u is harmonic in $\mathcal{S}$, and being piecewise linear is globally linear. ⋄

Thanks to the previous result, we can introduce the space

$$V_{hH} := V_{0h} \oplus V_{0H}. \tag{8}$$

As we decided to identify functions of V_{0h} and of V_{0H} with their extension by zero to the whole Ω, every function w_{hH} in V_{hH} can be written, in a unique way, as $w_{hH} = u_h + v_H$ with $u_h \in V_{0h}$ and $v_H \in V_{0H}$.

4. QUADRATURE

We are going to introduce now the numerical integration formula to be used in (5). Recall that the quadrature formula with integration points at the vertices is exact for polynomials of degree less than or equal to one. In particular, for a given triangle $\hat{T}$ one has

$$\int_{\hat{T}} g\, dxdy \approx \frac{|\hat{T}|}{3} \sum_{i=1,2,3} g(q_i) \quad \forall g \in P_1(\hat{T}). \tag{9}$$

Hence we introduce the following quadrature rule.

$$(\nabla u, \nabla v)_{hH} := \sum_{T\in\mathcal{T}_h} \frac{|T|}{3} \sum_{i=1,2,3} \frac{\nabla(u_{|T})\cdot\nabla v}{I_{\Omega_1}+I_{\Omega_2}}|_{q_i(T)}$$

(10)

$$+ \sum_{K\in\mathcal{K}_H} \frac{|K|}{3} \sum_{j=1,2,3} \frac{\nabla(v_{|K})\cdot\nabla u}{I_{\Omega_1}+I_{\Omega_2}}|_{q_j(K)}.$$

where $I_\Omega(x) = 1$ if $x \in \Omega$ and zero otherwise.

Remark 2. *Notice that the notation $g|_{q_i(T)}$ is used with the following meaning. By assumption g is constructed from g_1, g_2 defined on $\mathcal{T}_h$ and $\mathcal{K}_H$ where g_1 (resp g_2), restricted to T (resp K), is smooth. This implies in particular that the restriction of g_1 to T has a unique extension to $\bar{T}$. Hence we take the restriction of g_1 to T first, and then we compute its value at the vertex $q_i(T)$. Due to our hypothesis, every vertex $q_i(T)$ is internal to some $K \in \mathcal{K}_H$, and hence $g(q_i(T))$ has a well defined meaning.*

Remark 3. *The notation $\nabla(u_{|T})$ is used to indicate that we first restrict the function u to T, and then we compute its gradient (which is actually constant in T) and similarly for $\nabla(v_{|K})$.*

We see here that if each vertex in strictly inside a triangle of the other triangulation there is no ambiguity. Therefore if it was not the case, moving slightly the vertices amount to choosing arbitrarily one quadrature formula and since there is no constant in the proof that follows which depends on the distance of vertices from the edges of the other triangulation we see that the hypothesis is purely formal.

The quadrature formula is obviously of order one for smooth functions and so by Strang's lemma the method will converge when h, H tend to zero provided that the bilinear form in (10) is coercive.
In the next Section we are going to prove that the integration formula (10) gives rise to a norm in the space V_{hH}, equivalent to the usual norm in $H^1_0(\Omega)$.

5. Ellipticity with numerical integration

In this section, for notational convenience, we are going to denote the generic function w in V_{hK} as $w = u - v$, with $u \in V_{0h}$ and $v \in V_{0H}$ (rather than using $w = u + v$ as in the previous section.) We start then by introducing, for $w = u - v \in V_{hH}$, the expression

$$|w|_{1,*} \equiv |u-v|^2_{1,*} := h_M^2 \sum_{T\in\mathcal{T}_h} \sum_{i=1,3} |\nabla(u_{|T}) - \nabla v|^2(q_i(T))$$

(11)

$$+ H_M^2 \sum_{K\in\mathcal{K}_H} \sum_{j=1,3} |\nabla u - \nabla(v_{|K})|^2(q_j(K)).$$

The notation has to be intended as in Remarks 2 and 3 It is clear that $(\nabla(u - v), \nabla(u-v))_{hH}$ can be bounded (from above and from below) by $|u-v|^2_{1,*}$, with constants independent of h_M and H_M. We are now going to show that on the space

V_{hH} they are both equivalent to $||\nabla(u-v)||^2_{L^2(\Omega)}$. Our aim is therefore to prove that

$$|u-v|_{1,*} \geq C||\nabla(u-v)||_{L^2(\Omega)} \tag{12}$$

where C depends only on C_T and C_K (as the converse inequality is trivial).

In order to prove (12) we need some further notation.

For every internal node q_T^r in $\mathcal{T}_h$ we define $m_T(\nabla u; r)$ as the arithmetic average of the values that ∇u assumes in the triangles T having q_T^r as a vertex. We *do not* take into account the measures of the different triangles. A similar meaning holds for $m_K(\nabla v; s)$.

An elementary computation shows that, if $T_1, ...T_k$ are the triangles having q_T^r as a vertex, then

$$\sum_{\ell=1}^{k} |\nabla(u_{T_\ell}) - \nabla v(q_T^r)|^2 = \sum_{\ell=1}^{k} |\nabla(u_{T_\ell}) - m_T(\nabla u; r)|^2 + k\,|m_T(\nabla u; r) - \nabla v(q_T^r)|^2. \tag{13}$$

To prove (13) one can use several geometric interpretations. However, the shortest way to describe the proof is just by brute force: if $a_1, a_2, ..., a_k$ and b are real numbers, and $m := (1/k)\sum a_\ell$, then use the obvious fact that $\sum(a_\ell - m) = 0$ to check that

$$\sum_{\ell=1}^{k}(a_\ell - b)^2 = \sum_{\ell=1}^{k}(a_\ell - m)^2 + k(m-b)^2.$$

It will also be convenient to introduce the notation

$$\sigma_T(\nabla u, r) := (\sum_{\ell=1}^{k(r)} |\nabla(u_{|T_\ell}) - m_T(\nabla u; r)|^2)^{1/2} \tag{14}$$

$$\sigma_K(\nabla v, s) := (\sum_{\ell=1}^{k(s)} |\nabla(v_{|K_\ell}) - m_K(\nabla v; s)|^2)^{1/2} \tag{15}$$

where obviously $k(r)$ and $k(s)$ are the number of elements having q_T^r (respectively, q_K^s) as a vertex.

Rearranging terms in (11) (that is, summing over the vertices instead of summing over the triangles) and using (13) (with the notation (14) on each vertex q_T^r, and (15) on each vertex q_K^s,) we obtain the following new expression for (11):

$$
\begin{aligned}
|u-v|_{1,*}^2 = h_M^2 \sum_{r=1}^{R}(|\sigma_T(\nabla u;r)|^2 + k(r)|m_T(\nabla u;r) - \nabla v(q_T^r)|^2) \\
+ H_M^2 \sum_{s=1}^{S}(|\sigma_K(\nabla v;s)|^2 + k(r)|m_K(\nabla s;s) - \nabla u(q_K^s)|^2).
\end{aligned} \tag{16}
$$

Before proving (12) we need some additional remark. In particular, we notice that, if T_1 and T_2 are two triangles, having an edge in common, and if q_T^r is one of the two common vertices, then

$$
|\nabla(u_{|T_1}) - \nabla(u_{|T_2})|^2 \leq 2|\sigma_T(\nabla u;r)|^2. \tag{17}
$$

With obvious notation, we have the analogue relation

$$
|\nabla(v_{|K_1}) - \nabla(v_{|K_2})|^2 \leq 2|\sigma_K(\nabla v;s)|^2. \tag{18}
$$

Our strategy to prove (12) is the following: we split the integral

$$
\int_\Omega |\nabla u - \nabla v|^2 dxdy
$$

as the sum on all the pieces Q (that can be written as the intersection of a triangle T with a triangle K) where both ∇u and ∇v are constants. The sum of the integrals over the pieces Q containing a vertex of one of the two decompositions is clearly bounded by $|\nabla u - \nabla v|_{1,*}$ (with constant 1), as written in the form (11), since the area of Q is obviously bounded by either h_M^2 or H_M^2, according to the type of vertex it contains. Hence we have to bound only the integrals

$$
\int_Q |\nabla u - \nabla v|^2 dxdy \tag{19}
$$

where $Q = T \cap K$ *does not contain any vertex*. It is clear that, in any case, the area of Q is bounded by h_M^2. The basic idea is then to identify a sequence of pieces $Q_1, Q_2, ..., Q_n$ such $Q_1 = Q$ and that any pair Q_i, Q_{i+1} has a piece of an edge of $\mathcal{T}_h$ in common, and Q_n contains a vertex (see the figure, where $n = 3$.)

If such a sequence is found, then we can denote by $T_1, T_2, .., T_n$ the elements of $\mathcal{T}_h$ such that $Q_i = T_i \cap K$, and by $q_T^{r_i}$ one of the two vertices of the edge common to T_i and T_{i+1}. Then we can write, using the triangle inequality and then (17) to bound each piece:

$$
\begin{aligned}
|\nabla(u_{|T}) - \nabla(v_{|K})| \leq \\
\leq \sum_{i=1}^{n-1} |\nabla(u_{|T_i}) - \nabla(u_{|T_{i+1}})| + |\nabla(u_{|T_n}) - \nabla(v_{|K})| \\
\leq \sqrt{2}\sum_{i=1}^{n-1} \sigma_T(\nabla u; r_i) + |\nabla(u_{|T_n}) - \nabla(v_{|K})|
\end{aligned} \tag{20}
$$

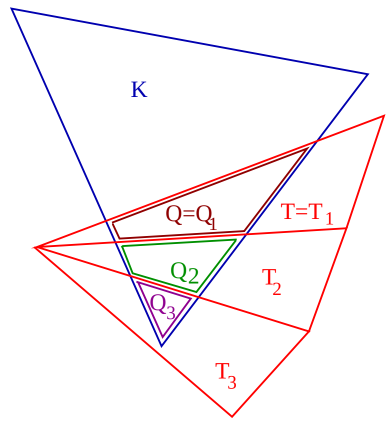

FIGURE 2. The sequence of polygons Q_i

Hence

$$\int_Q |\nabla(u_{|T}) - \nabla(v_{|K})|^2 dxdy \tag{21}$$
$$\leq h_M^2 C_n \Big(\sum_{i=1}^{n-1} \sigma_T^2(\nabla u; r_i) + |\nabla(u_{|T_n}) - \nabla(v_{|K})|^2\Big),$$

with C_n a constant depending only on the number n of steps in the sequence. Now, each of the pieces of the sum appearing in (21) can be bounded by (16), and the last one has already been bounded, since $T_n \cap K$ contains a vertex.

It is clear that we could also proceed by keeping T fixed, and changing K at each step; or keeping one of the two (T or K) fixed, and changing the other (K or T) at each step. It is not difficult to realize that in the above assumptions we have a uniform upper bound for the minimum necessary n (and hence for C_n), depending only on C_T and C_K. This can be easily seen if the two triangulations have comparable size: as each triangle cannot be too *thin* (thanks to (6)), there are only a limited number of triangles of one triangulation that can intersect a given triangle of the other triangulation. On the other hand, if h_M is much smaller than H_M, we have just to keep T fixed: a single (small) triangle T can intersect only a limited number of (bigger) triangles K.

Moreover (and this is more delicate,) the *number of times* that we are going to use (21) is bounded from above by the "number of vertices in $\mathcal{K}_H$ that belong to an element with non-empty intersection with Ω_1". Again, this is pretty easily seen if the two triangulations have comparable size. If not, that is if h_M is much smaller than H_M, we remark that a small triangle T, in order to intersect a bigger triangle K on a piece Q that does not contain any vertex, has to be sufficiently close (of the order of h_M!) to a vertex of K (see the figure.) Hence, only a limited number of such T's in this situation can be found for each given vertex in $\mathcal{K}_H$.

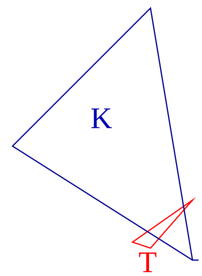

FIGURE 3. Two triangles which intersect but have no vertices inside one another

In conclusion, the sum of all the integrals of the type (19) can also be bounded by a fixed constant (depending only on C_T and C_K) times $|\nabla u - \nabla v|_{1,*}$, and (12) is proved.

Since $\left((\nabla u, \nabla u)_{hH}\right)^{1/2}$ is a norm on $V_{hH} = V_{0h} \oplus V_{0H}$, all classical results on the convergence of iterative schemes can be easily applied. For instance, let $\beta \geq 0$ be some positive scalar, let u^0, v^0 be arbitrary functions of V_{0h} and V_{0H} respectively, and consider the loop:

$$
(22)\quad \left|\begin{array}{l}
\text{find } \{u^{n+1}, v^{n+1}\} \in V_{0h} \times V_{0H} \text{ solution of} \\
\beta(u^{n+1} - u^n, \hat{u})_{hH} + (\nabla(u^{n+1} + v^n), \nabla\hat{u})_{hH} = (f, \hat{u})_{hH} \ \forall\hat{u} \in V_{0h} \\
\beta(v^{n+1} - v^n, \hat{v})_{hH} + (\nabla(v^{n+1} + u^n), \nabla\hat{v})_{hH} = (f, \hat{v})_{hH} \ \forall\hat{v} \in V_{0H}
\end{array}\right.
$$

It is clear that, for $\beta = 0$ this is a particular case of the abstract overlapping Schwarz method analysed in [7]. It is easy to see that the abstract results of [7] imply the geometric convergence of the algorithm for any fixed pair of decompositions, although some additional work would be needed to check whether the contraction constant stays uniformly away from 1 when the meshsizes h_M and H_M go to zero. On the other hand, for $\beta > 0$ the analysis of [3] of the algorithm (22) applies unchanged.

6. NUMERICAL TEST

Potential flow around an airfoil involves solving Laplace's equation in a domain outside the airfoil. The finite element method of order one on triangles has been used. The domain is divided in two: a domain near the airfoil which is triangulated with small triangles and the rest of the domain which uses bigger triangles. Here the domain has two airfoils, a large one and a small one. The decomposition must be such that the physical domain is the union of both domain and the domains must overlap. Then Schwarz algorithm is used with translation and quadratures at the vertices as explained above. Four iterations are sufficient for convergence to machine accuracy.

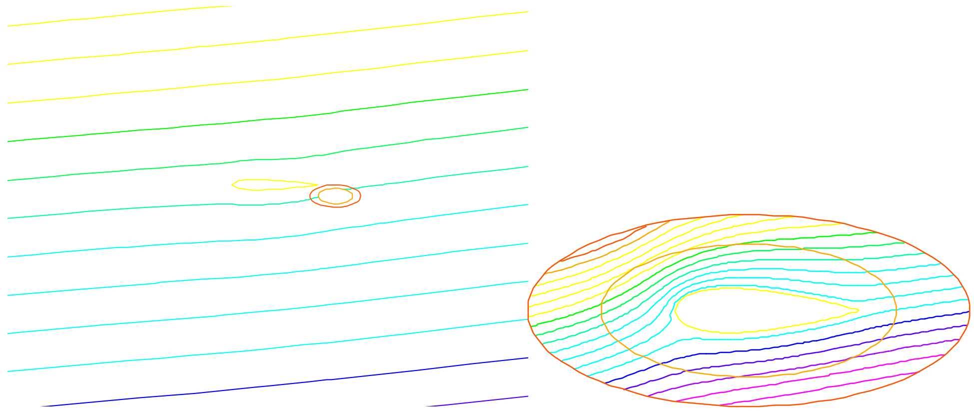

FIGURE 4. *Stream function around a two-pieces airfoil, namely solution of* $\Delta\psi = 0$ *with Dirichlet data by the Chimera method (i.e. Schwarz algorithm). The convergence is obtained after 4 iterations.*

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