

Mesh Adaption for the Black & Scholes Equations

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Abstract

We investigate the accuracy of mesh adaption for a problem with a free boundary that arises in finance for the pricing of American options. The formulation is given; it is discretized by the Finite Element Method (the connection with Finite Differences is recalled because FEM is not common in banking) and mesh adaption by the modified metric-Voronoi approach is presented and tested.

1 Introduction

The Black & Scholes equation is used in finance to price an option on the market.

Consider an American call option on an asset which is worth S_t dollars at time t .

We want to pay C dollars at time 0 to place an option which will give us the *right to buy* the asset at any time $\tau \in (0, T)$ for K dollars (the strike). We are not obliged to exercise our right to buy the asset but after time T the deal is void. Obviously if the asset is worth more than K at time T (i.e. if $S_T > K$) we will exercise it and if it is less we will not.

Furthermore if r is the interest rate of riskless commodities there will be a profit if $S_\tau > e^{r\tau}C + K$. The problem is to find C or more generally $C_t(x)$ the price of the call for all x, t .

Conversely as an owner we want to give P dollars to have the *right to sell* the asset S_τ at price K at time $\tau \leq T$. Obviously we will exercise the right to sell at T or before if $S_T < K$; the profit will be $K - S_\tau - Pe^{r\tau}$.

The price of the put at later time is denoted by P_t . The "no-arbitrage" hypothesis implies that

$$C_t + Ke^{-r(T-t)} = P_t + S_t \quad (1)$$

Furthermore the same no-arbitrage hypothesis tells us that an American Call will cease to exist if for some $\tau < T$, $C_\tau < S_\tau - K$; similarly an

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American Put will cease to exist if for some $\tau < T$, $P_\tau < K - S_\tau$. This leads to the constraints

$$C_t \geq S_t - K, \quad P_t \geq K - S_t. \quad (2)$$

1.1 Notations

So, through out the paper we will denote by

- t the time,
- x a variable which is destined to be the price S_t of the asset when it is used in conjunction with t ;
- S_t : the price of the asset follows a stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dB). \quad (3)$$

- μ : average tendency of the price of the asset per dollar
- σ : volatility of the asset
- $C(x, t)$: price of a call option on an asset of value x and at time t .
- $P(x, t)$: price of a put option on an asset of value x and at time t .
- $u(x, t)$: price of an option (put or call) on an asset of value x and t .
- r : risk free interest rate.
- $\varphi(x)$ the value of the option at time T .

2 The Black and Scholes equation

The computation of u in the model of Black & Scholes involves the solution of the following parabolic equation with given final data:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0 \quad \text{in } R^+ \times]0, T[\quad (4)$$

This is a consequence of the fact that C_t and P_t are given functions of S_t and that the expected value of S_t satisfies (4), a well known property of the stochastic differential equation (3).

2.1 Boundary Conditions

At time T the price of the option is the profit made by realizing the option,

$$C(x, T) = \varphi(x) \equiv (x - K)^+ \quad P(x, T) = (K - x)^+ \quad (5)$$

We know also that the model should give $0 < C < x$ because the option must be cheaper than the asset; that gives a boundary condition, $C = 0$ at $x = 0$. But because of the singularity at $x = 0$ of the coefficients of the PDE, if u is regular near $x = 0$, (i.e. $\partial_{xx}u$ bounded) the PDE contains a hidden boundary condition (the limit of the PDE at $x = 0$):

$$\frac{\partial u}{\partial t} - ru = 0 \quad \text{at } x = 0 \quad (6)$$

i.e. if r is constant:

$$u(0, t) = u(0, T)e^{-\int_t^T r(\tau, 0)d\tau} = u(0, T)e^{r(t-T)} \quad (7)$$

which in the case $u(0, T) = C(0, T) = 0$ gives $C(0, t) = 0$ for all time.

At infinity the PDE contains also a boundary condition embedded in the hypothesis that the solution be regular (see Nicolaides[8]). It seems numerically more appropriate to impose

$$\lim_{x \rightarrow \infty} [u(x, t) - \varphi(x)e^{r(T-t)}] = 0 \quad (8)$$

because it is compatible both with the final condition at T and with the Put-Call relation (1).

We will impose this condition not at $x = +\infty$ but at $x = L$, a process which is called "localization" in numerical finance

$$u(L, t) = \varphi(L)e^{r(T-t)} \quad (9)$$

However a "non-reflective" boundary condition would probably be more efficient.

2.1.1 Change of variable

To remove the singularity at $x = 0$ the following change of variable is proposed

$$u(y, t) = C(e^y, t) \quad (10)$$

$$\mu' = \mu - \frac{1}{2}\sigma^2 \quad (11)$$

$$\tau = T - t \quad (12)$$

Then the problem becomes

$$\begin{aligned} \frac{\partial \varphi}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 \varphi}{\partial y^2} - \mu' \frac{\partial \varphi}{\partial y} + r\varphi &= 0 \quad \text{in } R \times]0, T[\\ u(y, 0) &= \varphi(e^y) \end{aligned} \quad (13)$$

and when R is approximated by $] -L, +L[$ then (2),(6) imply

$$\begin{aligned} u(L, \tau) &= (e^L - K)e^{r\tau}, \quad \text{i.e. } C(x, T-t) \simeq xe^{r(T-t)} \quad \text{when } x \gg 1, \\ u(-L, \tau) &= \varphi(0)e^{r\tau} \end{aligned} \quad (14)$$

2.2 Existence of solution

It is quite easy to show that (13)14 has a unique solution and deduce from the previous change of variable that (4) has also a unique solution; however it is not easy to see if there is a singularity at $x = 0$ or not, and this is numerically important.

Notice however that if (4) was solved for R instead of just R^+ the restriction to R^+ would be the correct solution because the equation gives an odd solution in x for odd boundary and initial data and because of this

hidden boundary condition at $x = 0$ which is imbedded in the equation. Now on this augmented domain $(-L, L) \times (0, T)$ Oleinik's theorem [9] tells that (4) with data in C^k at $t = T$ and smooth Dirichlet data at $x = \pm L$ has unique solution in $L^2(0, T, H^k(-L, L))$. Therefore

Proposition Equation (4) on $(0, L) \times (0, T)$ with (7), (9) and final data in C^k has a unique solution in $L^2(0, T, H^k(0, L))$.

Unfortunately the final data (5) have the C^0 regularity only; but it is well known that diffusion has a regularizing effect so that at $t < T$ the solution is regular. Although this is only a heuristic argument it gives a good indication that u_{xx} is regular at $x = 0$ at all time. This is an important information because then there is no reason to use the change of variable numerically as it concentrates the grid points around $x = 0$ unnecessarily.

Besides, $y = e^x$ is a frightening change of variable because y may go beyond the range of numbers allowed by the compiler and it is justified only if the coefficients σ, r are constant and perhaps also only for semi-academic studies.

2.2.1 Stability

Equation (13) multiplied by u and integrated gives

$$\int_{\Omega} \frac{1}{2} \frac{\partial u^2}{\partial \tau} + \int_{\Omega} \frac{u^2}{2} \frac{\partial \mu'}{\partial y} + \int_{\Omega} \left(\frac{\partial u}{\partial y} \right)^2 \frac{\sigma^2}{2} - \int_{\Omega} \frac{u^2}{2} \frac{\partial^2 \sigma^2}{\partial y^2} + \int_{\Omega} r u^2 = \int_{\partial \Omega} \dots \quad (15)$$

The "energy" $E = \int_{\Omega} u^2$ will decay with time if

$$\frac{d\mu}{dy} - \frac{1}{2} \frac{d^2 \sigma^2}{dy^2} + r \geq 0 \quad (16)$$

because (15) gives a negative sign to $\partial E / \partial \tau$.

Remark

A change of variable shows that this hypothesis is not essential.

Let $u_1 = e^{-\alpha \tau} u$ then (13) becomes

$$\frac{\partial u_1}{\partial \tau} + \alpha u_1 - \mu' \frac{\partial u_1}{\partial y} - \frac{\sigma^2}{2} \frac{\partial^2 u_1}{\partial y^2} + r u_1 = 0 \quad (17)$$

so r is changed into $r + \alpha$

Numerically however if (16) is not verified it is a good idea to do this change of variable because it removes the exponential growth of u , something which is always difficult to capture because of the "overflow" of real numbers when the result of an arithmetic operation is too large.

2.3 Discretization with finite differences

The implicit Euler scheme with centered spatial differences is

$$\begin{aligned} & \frac{1}{k}[u_j^{m+1} - u_j^m] - \frac{1}{2} \frac{\sigma_j^{m+1}}{h^2} [u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}] \\ & - \frac{\mu_j^{m+1}}{2h} (u_{j+1}^{m+1} - u_{j-1}^{m+1}) + \frac{r_j^{m+1}}{2} u_j^m = 0 \end{aligned} \quad (18)$$

It can be applied also to the system written in (x, t) by exchanging the role of m and $m + 1$ as the scheme must go backward in time .

In the case of (17) the coefficients σ and μ' are constant and the analysis of Von Neumann shows stability if μ' is small enough; if not upwinding must be applied and $u_{j+1}^m - u_{j-1}^m$ replaced by $2(u_{j+1}^m - u_j^m)$ (recall that $\mu' > 0$).

A better scheme, widely used in finance is the Crank-Nicolson scheme (see [6] and [7]) also known as the theta-scheme with $\theta = 1/2$. But for our purpose of comparison with the finite element method we will use (18) for clarity, without loss of generality.

2.4 Discretization by FEM

Consider the B&S equation without change of variable but localized on a finite interval $(0, L)$:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + r x \frac{\partial u}{\partial x} - r u = 0 \quad \text{in } (0, L) \times]0, T[\quad (19)$$

Assume that $u(0, t) = 0$, $u(L, t)$ and $u(x, T)$ are given, and that $u(L, t) = u(L, T) e^{-r(T-t)}$.

Let us discretized in time by an implicit Euler scheme backward in time

$$\frac{1}{\delta t} (u^{m+1} - u^m) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u^m}{\partial x^2} + r x \frac{\partial u^m}{\partial x} - r u^m = 0 \quad (20)$$

It is easy to show that for $u = u^m$ this equation in variational form is

$$\int_0^L (\alpha u w + \frac{1}{2} \sigma^2 x^2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} - \mu x \frac{\partial u}{\partial x} w) dx = \int_0^L f w dx \quad \forall w \in V \quad (21)$$

for some appropriate α, μ and f .

Here we may take for V the weighted norm Sobolev space

$$V = \{w : w, x \frac{\partial w}{\partial x} \in L^2(0, L), w(L) = 0\} \quad (22)$$

which is a Hilbert space with the scalar product

$$\langle u, v \rangle = \int_0^L (uv + x^2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}).$$

It is also straightforward to show that the bilinear form $a(u, w)$ defined on V by the left hand side of the variational equation is coercive and continuous because $\sigma > 0$.

Then the Lax-Milgram Lemma tells us that there is one and only one solution to this problem because the right hand side is continuous.

Notice that it is not necessary to impose a boundary condition at $x = 0$ because it is embedded in the equation and in the hypothesis $x \rightarrow xu_x$ square integrable.

Let us discretize the problem by the Galerkin method by taking a finite dimensional approximation of V :

$$V_h = \{v_h \in u^0(0, L) : v_h|_{(x_i, x_{i+1})} \in P^1, \quad v_h(L) = 0\} \quad (23)$$

where the x_i are such that $\cup_i(x_i, x_{i+1}) = (0, L)$.

A basis for V_h is the hat functions of all but the first and last x_i :

$$w^i(x) = \begin{cases} 0 & \text{if } x < x_{i-1} \\ (x - x_{i-1})/(x_i - x_{i-1}) & \text{if } x_{i-1} < x < x_i \\ (x_{i+1} - x)/(x_{i+1} - x_i) & \text{if } x_i < x < x_{i+1} \\ 0 & \text{if } x_{i+1} < x \end{cases} \quad (24)$$

Knowing that $u_h(x) = u_i w^i(x) + u_{i+1} w^{i+1}(x)$ on (x_i, x_{i+1}) it is straightforward to compute $a(u_h, w^j)$ and get the finite difference formulation associated with this method. For example with $\mu = 0$, the implicit scheme is

$$\begin{aligned} & \frac{1}{8\delta t} [(u_{i+1}^{n+1} - (1 + \frac{r}{dt})u_{i+1}^n)(x_{i+1} - x_i) + 4(u_i^{n+1} - (1 + \frac{r}{dt})u_i^n)(x_{i+1} - x_{i-1}) \\ & + (u_{i-1}^{n+1} - (1 + \frac{r}{dt})u_{i-1}^n)(x_i - x_{i-1})] \\ & - \frac{\sigma^2}{6} [u_{i+1}^n (\frac{x_{i+1}^3 - x_i^3}{(x_{i+1} - x_i)^2}) + u_i^n (\frac{x_{i+1}^3 - x_i^3}{(x_{i+1} - x_i)^2} + \frac{x_i^3 - x_{i-1}^3}{(x_i - x_{i-1})^2}) \\ & + u_{i-1}^n (\frac{x_i^3 - x_{i-1}^3}{(x_i - x_{i-1})^2}) = 0 \end{aligned} \quad (25)$$

3 American Options

The model now requires that $u(x, \tau)$ never becomes larger (or smaller) than $\psi(x, \tau)$ given. Thus it is a time dependent variational inequality

The problem is,

$$\min\{\frac{\partial u}{\partial \tau} - \mu' \frac{\partial u}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + ru; u - \psi\} = 0 \quad (26)$$

3.1 Discretization in time

Applying the implicit Euler scheme leads to

$$\min\{\frac{u^{n+1} - u^n}{\delta t} + \mu' x \frac{\partial u^n}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 u^n}{\partial x^2} - ru^n, u^n - \psi\} = 0 \quad (27)$$

which, at each time step, is a problem of the type

$$\min\{Au^n - d, u^n - \psi\} = 0 \quad (28)$$

3.1.1 The Brennan-Schwartz algorithm

A projection algorithm would compute

$$Au^{n+1/2} = d \quad (29)$$

and set

$$u_i^n = \max(u_i^{n+1/2}, \psi_i) \quad (30)$$

at each grid points y_i .

The Brennan-Schwartz algorithm combines this projection algorithm with a partial solution of the linear system replaced by

$$\tilde{A}u^{n+1/2} = (\tilde{A} - A)u^{n+1} - d \quad (31)$$

where \tilde{A} corresponds to one iteration of a Gauss-Seidel relaxation step.

4 Multidimensional Black and Scholes equation

4.1 General form

When there are d assets, $x = (S_1, S_2, \dots, S_d)$ is multidimensional and all vectors are in R^d . Then $\mu\partial u/\partial x$ is replaced by $\vec{\mu} \cdot \nabla u$, the Laplace operator $-\Delta$ replaces $-\partial^2 u/\partial x^2$ and σ^2 becomes $\sigma\sigma^T$ with a matrix σ . The stochastic ODE for each asset is

$$dS_i = S_i(\mu_i dt + \sigma_i dW_i) \quad (32)$$

but the Weiner processes are correlated by

$$\langle dW_i dW_j \rangle = \rho_{ij} dt, \quad \rho_{ij} \in (-1, 1) \quad (33)$$

so that the generalized Black and Scholes equations is

$$\partial_t u + \frac{1}{2} \sum_{i,j=1}^d \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum r x_i \frac{\partial u}{\partial x_i} - ru = 0. \quad (34)$$

which can also be written in divergence form as (recall that $\rho_{ij} = \rho_{ji}$)

$$\partial_t u + \frac{1}{2} \sum_{i,j=1}^n \sigma_i \sigma_j \rho_{ij} \frac{\partial}{\partial x_i} (x_i S_j \frac{\partial u}{\partial x_j}) + \sum x_i \frac{\partial u}{\partial x_i} (r - \sigma_i \sum_j \sigma_j \rho_{ij}) - ru = 0. \quad (35)$$

An option on two assets leads to a Black-Scholes equations in two space variables, For example in Jarrow[5] or Wilmott[10].

$$\partial_t u + \frac{(\sigma_1 x_1)^2}{2} \frac{\partial^2 u}{\partial x_1^2} + \frac{(\sigma_2 x_2)^2}{2} \frac{\partial^2 u}{\partial x_2^2}$$

$$+ \rho x_1 x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + r S_1 \frac{\partial u}{\partial x_1} + r S_2 \frac{\partial u}{\partial x_2} - r P = 0 \quad (36)$$

which is to be integrated in $(0, T) \times R^+ \times R^+$ subject to, in the case of a put

$$u(x_1, x_2, T) = (K - \max(x_1, x_2))^+. \quad (37)$$

Boundary conditions for this problem may not be so easy to device.

As in the one dimensional case the PDE contains boundary conditions on the axis $x_1 = 0$ and on the axis $x_2 = 0$, namely two one dimensional Black-Scholes equations driven respectively by the data $u(0, +\infty, T)$ and $u(+\infty, 0, T)$. These will be automatically accounted for because they are embedded in the PDE. So if we do nothing in the variational form (i.e. if we take a Neuman boundary condition at these two axis in the strong form) there will be no disturbance to these.

At infinity in one of the variable, as in 1D, it makes sense to match the final condition:

$$u(x_1, x_2, t) \approx (K - \max(x_1, x_2))^+ e^{r(T-t)} \quad \text{when } |x| \rightarrow \infty. \quad (38)$$

For an American put we will also have the constraint

$$u(x_1, x_2, t) \geq (K - \max(x_1, x_2))^+ e^{r(T-t)}. \quad (39)$$

5 Mesh adaption with triangular finite elements

5.1 Automatic triangulation of a square

5.1.1 The problem

Assume that we are given $N+4$ points the last four of which defines a square containing all the other points. The points are numbered $(q^0, \dots, q^{N-1}, q^N, q^{N+1}, q^{N+2}, q^{N+3})$

5.1.2 Algorithm

We maintain a list of triangles. At start it is the two triangles $(q^N, q^{N+1}, q^{N+2}), (q^{N+2}, q^{N+3}, q^N)$.

Then we perform a loop on the points, backward for simplicity:
for $i=N-1$ down to 0 do

- case 1: there exists a triangle of the list which contains q^i strictly. then replace this triangle by the 3 subtriangles which have q^i for vertex.
- case 2: the point q^i is on the border of the square. Then find the unique triangle which contains q^i and replace it by the 2 subtriangles which have q^i for vertex.
- case 3: There are two triangles which contain q^i (i.e. q^i is on an inner edge. Then each triangles must be replaced by the two sutriangles which have q^i for vertex..

5.1.3 The Delaunay criteria

The previous triangulation is admissible for the finite element method but it is a bad one, it "looks terrible". By this we mean that there are many obtuse angles and small triangles near to large ones.

Notice that to each inner edge of a triangulation we can associate a quadrangle made of the two triangles adjacent to the edge.

A triangulation is said to be "Delaunay" if for each edge the circle circumscribing one triangle does not contain the fourth vertex.

Edge swap If the 4 vertices of a quadrangle associated to an inner edge are not cocyclic then the two configurations obtained by swapping diagonals in the quadrangle, one is Delaunay, the other is not.

Proposition When a configuration becomes Delaunay by an edge swap the minimum angle in the 2 triangles increase.

Algorithm

Loop until nothing changes

Loop on the inner edges E

- Find the 2 triangles T_k, T_l adjacent to E; denote q^1, q^2, q^3 the vertices of T_k and by q^2, q^1, q^4 those of T_l .
- Check the Delaunay criteria. If it fails replace T_k, T_l by the triangles q^3, q^4, q^1 and q^4, q^3, q^2

Proposition *The algorithm converges.*

Indeed at each loop the smallest angle increase. When it no longer increases then the next to smallest angle increase.... The number of configurations being finite the process converges. The complexity of the algorithm is $O(N)$ (See [3] for more details).

5.1.4 Generation of interior points

In practice we use the previous algorithm without interior points; all vertices are on the boundary. We assume that the user has input his request on vertex density through the density of points on the boundary.

So each vertex has a weight. For boundary vertices it is the average length of the two surrounding boundary edges.

Then we perform the following test on each edge of the triangulation:

- the length of an edge is larger than the average weight of its vertices then we divided the edge by adding a middle point and assigning to it the average of the weights of the two vertices of the edge.

Then the triangulation algorithm is applied again to the new set of points.

And so on till no edge is divided.

5.2 Mesh adaption to a function

Mesh adaption is an important asset of unstructured mesh solvers. In [4] an adaptive procedure is presented, based on a change of metric in the Delaunay - Voronoi algorithm.

The idea is that the error of interpolation on a mesh is bounded by

$$\|u - u_h\| < C \|\nabla(\nabla u)\| h^2$$

where $\nabla(\nabla u)$ is the Hessian matrix of u . Therefore an attempt to keep $\vec{h}^T \nabla(\nabla u) \vec{h}$ constant is likely to work and build an adapted anisotropic mesh. If several functions are specified as for the mesh adaption, for instance u and v then it is $\min\{\vec{h}^T \nabla(\nabla u) \vec{h}, \vec{h}^T \nabla(\nabla v) \vec{h}\}$ which will be kept constant and equal to ϵ . More precisely the method is to apply the Delaunay-Voronoi triangulation algorithm with the distance based on these Hessians (so that circles become ellipses). It is however substantially more complex because a function may have a non positive definite Hessian.

To define a metric the Hessian is diagonalized:

$$\nabla(\nabla u) = \begin{pmatrix} \partial^2 u / \partial x^2 & \partial^2 u / \partial x \partial y \\ \partial^2 u / \partial x \partial y & \partial^2 u / \partial y^2 \end{pmatrix} = \mathcal{R} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathcal{R}^{-1},$$

where R is the eigenvectors matrix of $\nabla(\nabla u)$ and λ_i its eigenvalues. The metric is defined by the scalar product $x^T \mathcal{M} y$, $x, y \in R^2$ with \mathcal{M} defined by:

$$\mathcal{M} = \mathcal{R} \begin{pmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{pmatrix} \mathcal{R}^{-1}, \text{ and } \tilde{\lambda}_i = \min(\max(\frac{|\lambda_i|}{\epsilon}, \frac{1}{h_{max}^2}), \frac{1}{h_{min}^2}),$$

with h_{min} and h_{max} being the minimal and maximal edge lengths allowed in the mesh and ϵ the tolerance.

The parameter ϵ is left at the choice of the user.

6 Numerical test

In freefem and freefem+ (a general PDE solver available on the web) mesh adaption is implemented and so it was easy to test the method for the two dimensional American put option described above, with the constants

$$\sigma_1 = 0.3, \quad \sigma_2 = 0.3, \quad \rho = 0.3, \quad r = 0.05, \quad K = 40, \quad T = 0.5 \quad (40)$$

An implicit Euler scheme with projection is used and a mesh adaption is done every 10 time steps. The first order terms are treated by the Characteristic Galerkin method, which, schematically, approximates

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} \approx \frac{1}{\delta t} (u^{n+1}(x) - u^n(x - \vec{a} \delta t)) \quad (41)$$

The listing of the freefem program is given below. The program is self-explanatory and gives all the numerical values needed to reproduce the test.

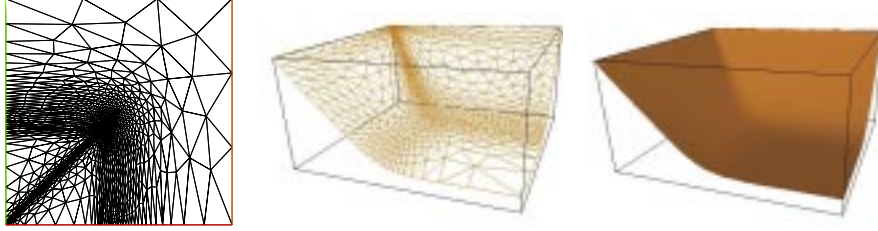


Figure 1: The mesh after 3 adaptions displayed flat and on the surface of the solution u . On the right the solution is displayed with shades as the surface $x_1, x_2, u(x_1, x_2)$. Notice that it captures quite well the singularities of the solution, namely the position of the free boundary and the line $x_1 = x_2$.

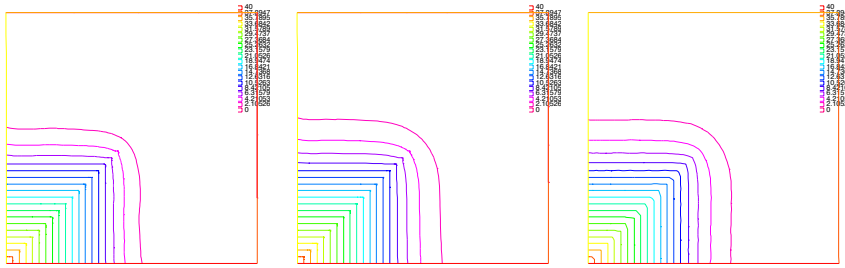


Figure 2: The level lines of u are displayed at $t = 0.5$ (left) for an adapted mesh (2 adaptions) and $t = 1$ for a refined mesh (center) with 4 adaptions and a random mesh. All 3 meshes have around 1500 vertices.

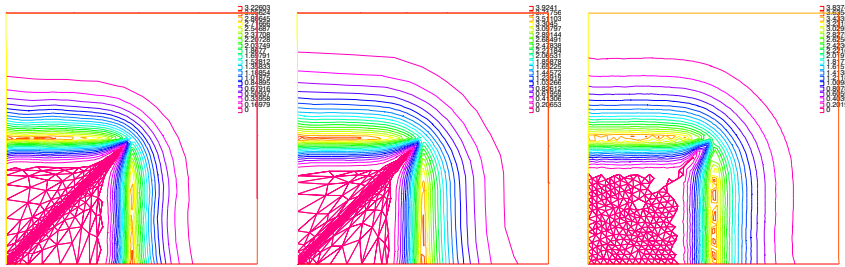


Figure 3: The level lines of $U - \max(K - \max(x_1, x_2))$. Where the triangles are seen, it means it is zero. The free boundary is the first line next to the triangles. On the left it is for $t = 0.5$ In the center and on the right for $t = 1$, the right figure being with a non-refined random mesh

```

wait:=0; m:=20; L:=80; LL:=80;
border aa(t=0,L){x=t;y=0};
border bb(t=0,LL){x=L;y=t};
border cc(t=L,0){x=t ;y=LL};
border dd(t=LL,0){x = 0; y = t};
mesh th = buildmesh(aa(m)+bb(m)+cc(m)+dd(m));
sigmax:=0.3; sigmay:=0.3; rho:=0.3; r:=0.05;
K=40; dt:=0.02;
f = max(K-max(x,y),0);
femp1(th) u=f;
      femp1(th) xveloc = -x*r+x*sigmax^2+x*rho*sigmax*sigmay/2;
      femp1(th) yveloc = -y*r+y*sigmay^2+y*rho*sigmax*sigmay/2;
j:=0;
for n=0 to 0.5/dt do
{
  solve(th,u) with AA(j){
    pde(u) u*(r+1/dt)
    - dxx(u)*(x*sigmax)^2/2 -dyy(u)*(y*sigmay)^2/2
    - dxy(u)*rho*sigmax*sigmay*x*y/2
    - dyx(u)*rho*sigmax*sigmay*x*y
    = convect(th,xveloc,yveloc,dt,u)/dt;
    on(aa,dd) dnu(u)=0;
    on(bb,cc) u = f;
  };
  u = max(u,f); plot("uf",th, u-f);
  if(j==10) then {
    mesh th = adaptmesh("th",th,u);
    femp1(th) xveloc = -x*r+x*sigmax^2+x*rho*sigmax*sigmay/2;
    femp1(th) yveloc = -y*r+y*sigmay^2+y*rho*sigmax*sigmay/2;
    femp1(th) u=u;
    wait:=0;   j:=-1;
  }; j=j+1;
};

```

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