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The Stability of Some Pexider Trigonometric Functional Equations

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Abstract

The aim of this paper is to investigate the stability problem of the pexider type trigonometric functional equations

$$f(2x) - f(2y) = 2g(x+y)f(x-y) \qquad (P_1)$$

$$g(2x) - g(2y) = 2f(x+y)f(x-y) \qquad (P_2)$$

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1. Introduction

In 1940, S.M.Ulam raised the following stability problem [5]:
Let f be a mapping from group G, to a metric group G' with metric $d(.,.)$ such that

$d(f(xy), f(x)f(y)) \leq \epsilon$. Then does there exists a group homomorphism L and $\delta_\varepsilon \geq 0$ such that $d\big(f(x), L(x)\big) \leq \delta_\varepsilon$ for all $x \in G$?

This problem was solved affirmatively by Hyers[3] under the assumption that is a Banach space. Baker et.al [1] introduced that if f satisfies the stability inequality $E_1(f) - E_2(f) \leq \varepsilon$, then either f is bounded or E_1(f)-E_2(f). The super stability of the cosine functional equation $f(x+y) + f(x-y) = 2f(x)f(y)$ and the sine functional equation $f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2$ are investigated by Baker [2].

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, and some exponential functions also satisfy the above mentioned equations; thus they can be called by the hyperbolic cosine (sine, trigonometric, exponential) functional equations, respectively. The super stability of the pexider type trigonometric functional equation has been studied by Kim [4].

The aim of this paper is to study the stability problem of the pexider type trigonometric functional equations

$$f(2x) - f(2y) = 2g(x+y)f(x-y) \tag{P_1}$$

$$g(2x) - g(2y) = 2f(x+y)f(x-y) \tag{P_2}$$

In this paper, let $(G, +)$ be an abelian group, C the field of complex numbers, R the field of real numbers. We may assume that f and g are non zero functions and ε is a non negative real constant, a mapping $\varphi: G \to R$.

2. Stability of the equation (P_1)

In this section, we investigate the stability of the functional equation (P_1).

Theorem 2.1 Suppose that $f, g: G \to C$ satisfy the inequality

$$|f(2x) - f(2y) - 2g(x+y)f(x-y)| \leq \varphi(x) \tag{2.1}$$

for all $x, y \in G$. If f fails to be bounded then g satisfy

$$g(2x) + g(2y) = 2g(x+y)g(x-y) \tag{2.2}$$

Proof: Equation (2.1) can be written as

$$|f(u+v) - f(u-v) - 2g(u)f(v)| \leq \varphi\left(\frac{u+v}{2}\right) \tag{2.3}$$

Let f be unbounded. Then we can choose a sequence $\{v_n\}$ in G such that $0 \neq |f(v_n)| \to \infty$ as $n \to \infty$. Taking $v = v_n$ in (2.3) we obtain

$$\left|\frac{f(u+v_n)-f(u-v_n)}{2f(v_n)}-g(u)\right| \le \frac{\varphi\left(\frac{u+v_n}{2}\right)}{2|f(v_n)|} \tag{2.4}$$

that is

$$\lim_{n\to\infty}\frac{f(u+v_n)-f(u-v_n)}{2f(v_n)} = g(u) \tag{2.5}$$

for all $u \in G$. Using (2.3) we have

$$\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right) \ge$$
$$|f(u+(v_n+v))-f(u-(v_n+v))-2g(u)f((v_n+v))|$$
$$+|f(u+(v_n-v))-f(u-(v_n-v))\,2g(u)f((v_n-v))| \tag{2.6}$$

so that

$$\left|\frac{f((u+v)+v_n)-f\big((u+v)-v_n)\big)}{2f(v_n)}+\frac{f((u-v)+v_n)-f\big((u-v)-v_n)\big)}{2f(v_n)}\right.$$
$$\left.-\,2g(u)\frac{f(v+v_n)+f(v-v_n))}{2f(v_n)}\right|$$
$$\le\ \frac{\varphi\left(\frac{u+v+v_n}{2}\right)+\varphi\left(\frac{u+v-v_n}{2}\right)}{2|f(v_n)|}$$

Taking lim as $n \to \infty$ on both sides

$$|g(u+v)+g(u-v)-2g(u)g(v)| \le 0$$

that is

$$|g(2x)+g(2y)-2g(x+y)g(x-y)| \le 0$$

for all $x, y \in G$. Therefore g satisfies (2.2).

Corollary 2.2 Suppose that $f, g: G \to C$ satisfy the inequality

$$|f(2x)-f(2y)-2g(x+y)f(x-y)| \le \varepsilon$$

for all $x, y \in G$. If f fails to be bounded then f and g satisfy

$$g(2x)+g(2y) = 2g(x+y)g(x-y)$$

3. Stability of the equation (P_2)

In this section, we investigate the stability of the functional equation (P_2).

Theorem 3.1 Suppose that $f, g: G \to C$ satisfy the inequality

$$|g(2x)-g(2y)-2f(x+y)f(x-y)| \le \varphi(x) \tag{3.1}$$

for all $x, y \in G$. If f fails to be bounded then f satisfy

$$f(2x) + f(2y) = 2f(x+y)f(x-y) \tag{3.2}$$

Proof: Equation (3.1) can be written as

$$|g(u+v) - g(u-v) - 2f(u)f(v)| \le \varphi\left(\frac{u+v}{2}\right) \tag{3.3}$$

Let f be unbounded. Then we can choose a sequence $\{u_n\}$ in G such that $0 \neq |g(v_n)| \to \infty$ as $n \to \infty$. Taking $v = v_n$ in (2.3) we obtain

$$\left|\frac{g(u+v_n) - g(u-v_n)}{2f(v_n)} - f(u)\right| \le \frac{\varphi\left(\frac{u+v_n}{2}\right)}{2|f(v_n)|} \tag{3.4}$$

that is $\lim_{n\to\infty} \frac{g(u+v_n)-g(u-v_n)}{2f(v_n)} = f(u)$ (3.5)

for all $u \in G$. Using (3.3) we have

$$\varphi\left(\frac{u+v+v_n}{2}\right) + \varphi\left(\frac{u+v-v_n}{2}\right)$$
$$\ge |g(u+(v_n+v)) - g(u-(v_n+v)) - 2f(u)f(v_n+v)| +$$
$$|g(u+(v_n-v)) - g(u-(v_n-v)) - 2f(u)f(v_n-v)| \tag{3.6}$$

so that

$$\left|\frac{g\big((u+v)+v_n\big) - g\big((u+v)-v_n\big)}{2f(v_n)} + \frac{g\big((u-v)+v_n\big) - g\big((u-v)-v_n\big)}{2f(v_n)} - 2f(u)\frac{f(v+v_n) - f(v-v_n)}{2f(v_n)}\right|$$
$$\le \frac{\varphi\left(\frac{u+v+v_n}{2}\right) + \varphi\left(\frac{u+v-v_n}{2}\right)}{2|f(v_n)|}$$

Taking lim as $n \to \infty$ on both sides

$$|f(u+v) + f(u-v) - 2f(u)f(v)| \le 0$$

where $f(v) \coloneqq \frac{f(v+v_n)-f(v-v_n)}{f(v_n)}$

that is $$|f(2x) + f(2y) - 2f(x+y)f(x-y)| \le 0$$

for all $x, y \in G$. Therefore f satisfy (3.2).

Corollary 3.2 Suppose that $f, g: G \to C$ satisfy the inequality

$$|g(2x) - g(2y) - 2f(x+y)f(x-y)| \le \varepsilon$$

for all $x, y \in G$. If f fails to be bounded then f satisfy

$$f(2x) + f(2y) = 2f(x+y)f(x-y).$$

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