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Formula for the Number of Spanning Trees in Light Graph

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Abstract

In this paper, we consider the outerplanar graph L_n[1], having 6n+6 vertices, 12n+9 edges and 6n+5 faces, in this graph all faces have degree 3 except for the outside face.
Our approach consists on finding a general formula that calculates the number of spanning trees in the Ligth graph L_n, depending on n.

Mathematics Subject Classification: 05C85, 05C30

Keywords: planar graphs, outer planar, spanning trees, ligth graph

1 Introduction

The number of spanning trees in a planar graph (network) is an important well-studied quantity and invariant of the graph; moreover it is also an important measure of reliability of a network which plays a central role in Kirchhoff's classical theory of electrical networks. In a graph (network), that contains several cycles, we must remove the redundancies in this network, i.e., we obtain a spanning tree. A spanning tree in a planar graph G is a tree which has the same vertex set as G (tree that passing through all the vertices of the map G)[2].
Generally, the number of spanning trees in a network can be obtained by computing a related determinant of the Laplacian matrix defined by $L(G) =$

$D(G) - A(G)$, with $D(G)$ and $A(G)$ are respectively the matrix of degrees and the adjacency matrix [3, 4]. However, for a large graph, evaluating the relevant determinant is computationally intractable. Wherefore, many works derive formulas to calculate the complexity for some classes of graphs. Bogdanowicz [7] derive the explicit formula $\tau(F_n)$; the number of spanning trees in F_n, A. Modabish and M. El Marraki investigated the number of spanning trees in the star flower planar graph [8], In [9] the authors proposed an approach for counting the number of spanning trees in the butterfly graph.
In the following, we describe a general method to count the number of spanning trees in the outerplanar light graph L_n [1], our work consist on combining several method as presented in [5], in order to calculate the complexity of L_n.

2 Preliminary Notes

An undirected graph is outerplanar if it can be drawn in the plane without crossings such that all vertices lie on the outerface boundary. That is, no vertex is totally surrounded by edges.

Let G_n the set of outerplanar graph shown in Figure 1. where v_1, s_1 and v_2 are the vertices of the outer face boundary of a plane network that delimite a number n of vertices which is the same between each pair of (v_1, s_1) and (s_1, v_2), also the total number of vertices of G_n is $|V_{G_n}| = 2n + 3$ and $n \geq 1$.

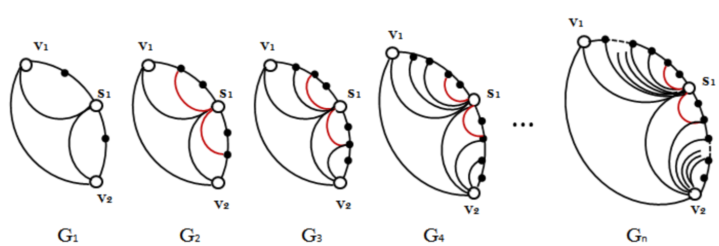

Figure 1: The family of graphs G_n

The connection of three G_n graphs lead to the outerplanar light graph shown in Figure 2.

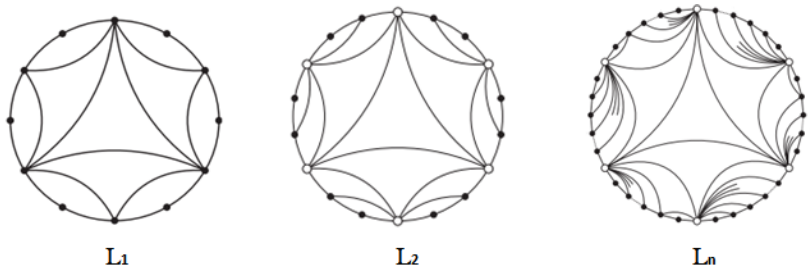

Figure 2: The outerplanar light graph L_n

Property 2.1 *The number of vertices, edges and faces of ligth graph satisfie respectively:* $|V_{L_n}| = |V_{L_{n-1}}| + 6 = 6(n+1)$, $|E_{L_n}| = |E_{L_{n-1}}| + 12 = 3(3+4n)$ *and* $|F_{L_n}| = |F_{L_{n-1}}| + 6 = 6n+5$.

Proof: The number of vertices of ligth graph is calculated recursively as:

$$|V_{L_n}| = |V_{L_{n-1}}| + 6 = \ldots = |V_{L_0}| + 6n = 6(n+1).$$

The same for edges and faces.
Before presenting the main results we need the following results:
Let G be the planar graph $G = G_1 \bullet G_2$ obtained by connecting G_1 and G_2 with a single vertex v_1 [5], then

$$\tau(G_1 \bullet G_2) = \tau(G_1) \times \tau(G_2). \tag{1}$$

Let G be a planar graph of type $G = G_1 | G_2$, (v_1 and v_2 two vertices of G_1 and G_2 connected by an edge e) [5], then

$$\tau(G) = \tau(G_1) \times \tau(G_2) - \tau(G_1 - e) \times \tau(G_2 - e). \tag{2}$$

Theorem 2.2 *[5] Let* $G = G_1 : G_2$ *be a planar graph,* v_1 *and* v_2 *two vertices of* G *which is formed by two planar graphs* G_1 *and* G_2*, then*

$$\tau(G) = \tau(G_1) \times \tau(G_2.v_1v_2) + \tau(G_1.v_1v_2) \times \tau(G_2).$$

Theorem 2.3 *[5, 6, 7] The number of spanning trees of the Fan* (F_n) *shown in the left of Figure 3, with* ($n = |V_{F_n}| - 2$) *satisfies:*

$$\begin{aligned} \tau(F_n) &= 3\tau(F_{n-1}) - \tau(F_{n-2}) \\ &= \frac{1}{\sqrt{5}} \Big((\frac{3+\sqrt{5}}{2})^{n+1} - (\frac{3-\sqrt{5}}{2})^{n+1} \Big), \; n \geq 1 \end{aligned}$$

3 Main Results

In this section we present some results to calculate the number of spanning trees in some families of outerplanar graphs to evaluate the complexity of light graph (L_n).

3.1 The complexity of X_n graph

X_n is the outerplanar graph illustrated on the right of Figure 3, having one vertex of degree n, 1 of degree 4, 2 vertices of degree 2 and the rest of degree 3, where $|V_{X_n}| = n + 2$.

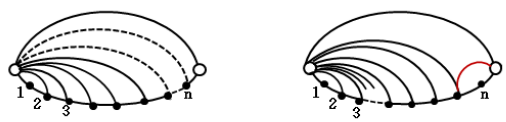

Figure 3: F_n and X_n graphs

Lemma 3.1 *The number of spanning trees of X_n is equal to the number of spanning trees of the n-fan:*

$$\tau(X_n) = \frac{1}{\sqrt{5}}\Big((\frac{3+\sqrt{5}}{2})^{n+1} - (\frac{3-\sqrt{5}}{2})^{n+1} \Big),\ n \geq 1$$

Proof: we apply equation (2)

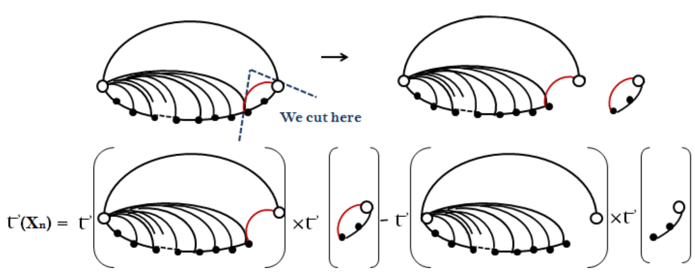

Figure 4: The complexity of X_n according to equation (2)

then:

$$\tau(X_n) = 3\tau(F_{n-1}) - \tau(F_{n-2}) = \tau(F_n)$$

3.2 The complexity of G_n graph

G_n is the outerplanar graph represented in Figure 1, with $|V_{G_n}| = 2n + 3$.

Lemma 3.2 *The number of spanning trees of the G_n graph depends on $\tau(F_n)$:*

$$\tau(G_n) = 2\tau(F_n)\Big(5\tau(F_{n+1}) - 12\tau(F_n)\Big)$$

Proof: By using equation (2) we get

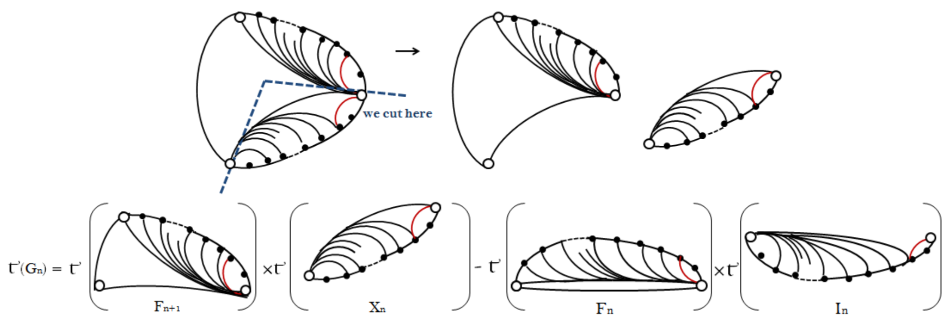

Figure 5: The complexity of G_n according to equation (2)

with:

$$\begin{cases} \tau(X_n) &= \tau(F_n) \quad \text{from Lemma 3.1} \\ \tau(I_n) &= 3\tau(F_{n-2}) \quad \text{from equation (1)} \end{cases}$$

then:

$$\tau(G_n) = \tau(F_n)\Big(\tau(F_{n+1}) - 3\tau(F_{n-2})\Big) \tag{3}$$

other hand:

$$\tau(F_{n+1}) = 3\tau(F_n) - \tau(F_{n-1}) \quad \Rightarrow \quad \tau(F_{n-1}) = 3\tau(F_n) - \tau(F_{n+1}) \tag{4}$$

and :

$$\tau(F_{n-2}) = 3\tau(F_{n-1}) - \tau(F_n) \quad \Rightarrow \quad \tau(F_{n-2}) = 8\tau(F_n) - 3\tau(F_{n+1}) \tag{5}$$

we replace $\tau(F_{n-2})$ of equation (5) in (3), then the result.

Corollary 3.3 *The complexity of the outerplanar graph G_n is given by the following formula:*

$$\tau(G_n) = \Big(\frac{6+4\sqrt{5}}{5}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^n + \Big(\frac{6-4\sqrt{5}}{5}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^n + \frac{18}{5}, \; n \geq 1$$

Proof: We use formula of Theorem 2.3 depending on n in Lemma 3.2, then the result.

3.3 The complexity of H_n graph

H_n is the outerplanar graph presented bellow in Figure 6:**a**, with $n = \frac{|V_{H_n}|-2}{2}$.

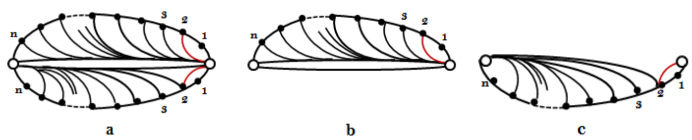

Figure 6: **a**: H_n graph, **b**: J_n graph and **c**: I_n graph.

Lemma 3.4 *The complexity of H_n depends on G_n and F_n:*

$$\tau(H_n) = \tau(G_n) - \tau(F_n)^2$$

Proof: From equation (2):

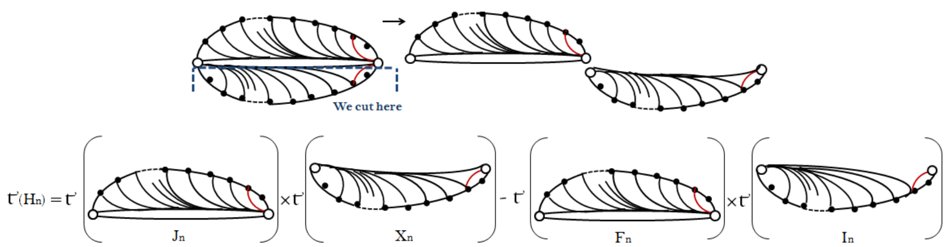

Figure 7: The complexity of H_n according to equation (2)

with :

$$\tau(J_n) = 2\tau(F_n) - \tau(F_{n-1})$$

then:

$$\tau(H_n) = \tau(F_n)\Big(2\tau(F_n) - \tau(F_{n-1}) - 3\tau(F_{n-2})\Big) \qquad (6)$$

we replace $\tau(F_{n-1})$ and $\tau(F_{n-2})$ in (6) using equations (4) and (5) of proof of Lemma 3.2, so:

$$\begin{aligned} \tau(H_n) &= \tau(F_n)\Big(2\tau(F_n) - \tau(F_{n-1}) - 3\tau(F_{n-2})\Big) \\ &= \tau(F_n)\Big(10\tau(F_{n+1}) - 25\tau(F_n)\Big) \\ &= \tau(G_n) - \tau(F_n)^2 \end{aligned}$$

with: $\tau(G_n) = \tau(F_n)\Big(10\tau(F_{n+1}) - 24\tau(F_n)\Big)$, from Lemma 3.2.

Corollary 3.5 *The complexity of the H_n graph is given by the following formula:*

$$\tau(H_n) = \Big(\frac{1+\sqrt{5}}{2}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^n + \Big(\frac{1-\sqrt{5}}{2}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^n + 4, \;\; n \geq 1$$

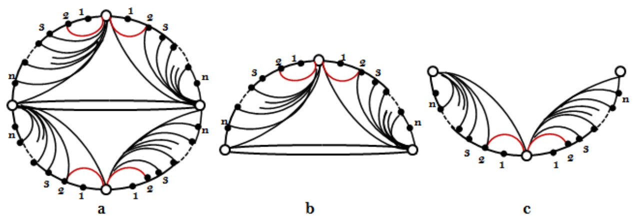

Figure 8: **a**: A_n graph, **b**: O_n graph and **c**: N_n graph

3.4 The complexity of A_n graph

A_n is the outerplanar graph illustrated in Figure 8:**a**, with $|V_{A_n}| = 4n + 4$.

Lemma 3.6 *The number of spanning trees of A_n is calculated by:*

$$\begin{aligned}\tau(A_n) &= 2\tau(G_n)\Big(\tau(G_n) - \tau(F_n)^2\Big)\\ &= -\Big(\frac{102+46\sqrt{5}}{25}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^{2n} - \Big(\frac{102-46\sqrt{5}}{25}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^{2n} + \Big(\frac{24-16\sqrt{5}}{25}\Big)\Big(\frac{3-\sqrt{5}}{2}\Big)^{2n}\\ &\quad +\Big(\frac{60+50\sqrt{5}}{5}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^{n} + \Big(\frac{306-234\sqrt{5}}{25}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^{n} + \Big(\frac{232+96\sqrt{5}}{25}\Big)\Big(\frac{47+21\sqrt{5}}{2}\Big)^{n}\\ &\quad +\Big(\frac{232-96\sqrt{5}}{25}\Big)\Big(\frac{47-21\sqrt{5}}{2}\Big)^{n} + \frac{116}{5},\ n \geq 1\end{aligned}$$

Proof: Again equation (2) implies that

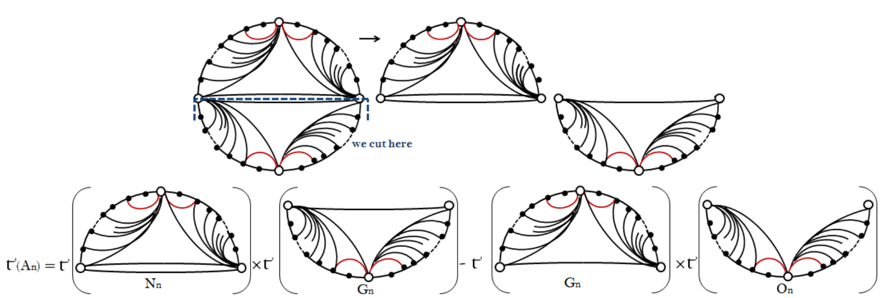

Figure 9: The complexity of A_n according to equation (2)

therefore: $\tau(A_n) = \tau(N_n)\times\tau(G_n) - \tau(G_n)\times\tau(O_n)$

with: $\begin{cases} \tau(N_n) &= 2\tau(G_n) - \tau(F_n)^2 \\ \tau(O_n) &= \tau(F_n)^2 \end{cases}$

then the result.

3.5 The complexity of light graph L_n

Results found previously allow us to calculate the number of spanning trees of light graph L_n; $n = \frac{|V_{L_n}|-6}{6}$.

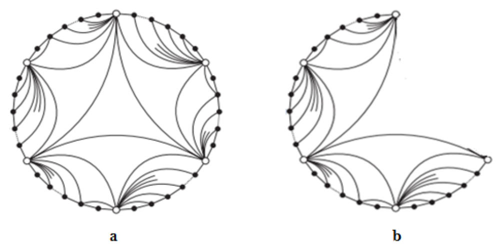

Figure 10: **a**: Light graph L_n and **b**: the M_n graph

Theorem 3.7 *The number of spanning trees in L_n is given by the following formula:*

$$\begin{aligned}
\tau(L_n) &= 3\tau(G_n)^2\Big(\tau(G_n) - \tau(F_n)^2\Big)\\
&= -\Big(\frac{5508+2484\sqrt{5}}{125}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^{2n} - \Big(\frac{5508-2484\sqrt{5}}{125}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^{2n}\\
&\quad +\Big(\frac{1296-864\sqrt{5}}{125}\Big)\Big(\frac{3-\sqrt{5}}{2}\Big)^{2n} + \Big(\frac{12876+9504\sqrt{5}}{125}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^{n}\\
&\quad +\Big(\frac{2316-1728\sqrt{5}}{125}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^{n} + \Big(\frac{19488+8064\sqrt{5}}{125}\Big)\Big(\frac{47+21\sqrt{5}}{2}\Big)^{n}\\
&\quad +\Big(\frac{19488-8064\sqrt{5}}{125}\Big)\Big(\frac{47-21\sqrt{5}}{2}\Big)^{n} + \Big(\frac{534+246\sqrt{5}}{25}\Big)\Big(161+72\sqrt{5}\Big)^{n}\\
&\quad +\Big(\frac{534-246\sqrt{5}}{25}\Big)\Big(161-72\sqrt{5}\Big)^{n} + \Big(\frac{924+396\sqrt{5}}{125}\Big)\Big(\frac{7+3\sqrt{5}}{2}\Big)^{2n}\Big(\frac{7-3\sqrt{5}}{2}\Big)^{n}\\
&\quad +\Big(\frac{924-396\sqrt{5}}{125}\Big)\Big(\frac{7-3\sqrt{5}}{2}\Big)^{2n}\Big(\frac{7+3\sqrt{5}}{2}\Big)^{n} + \frac{264}{5},\ n \geq 1
\end{aligned}$$

Proof: We cut L_n as shown in Figure 11

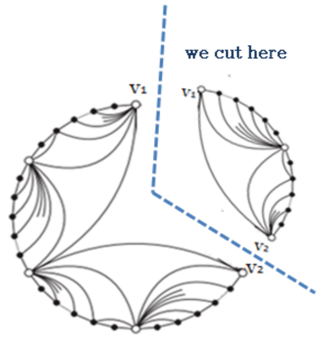

Figure 11: Light graph L_n after cutting

The two subgraphs have vertices v_1 and v_2 in common, so using Theorem 2.2 we get

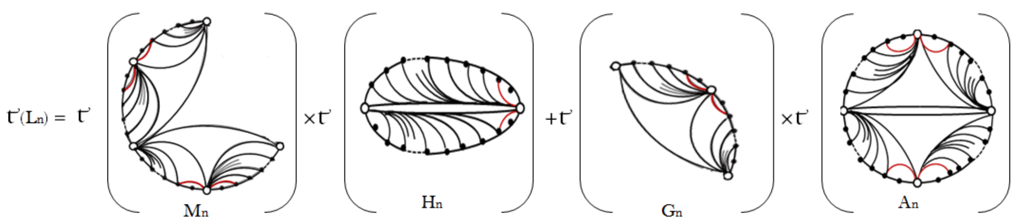

Figure 12: The complexity of L_n according to Theorem 2.2

so: $\tau(L_n) = \tau(M_n)\times\tau(H_n) + \tau(G_n)\times\tau(A_n)$
with: $\tau(M_n) = \tau(G_n)^2$ from equation (1), and using Lemma 3.6 and 3.4 we obtain the result.

References

[1] I. Fabrici, Light graphs in families of outerplanar graphs, *Discrete Mathematics,* **307** (2007), 866 - 872.

[2] D.B West. Introduction to graph theory. *second edition, prentice hall,* (2002).

[3] G. G. Kirchhoff. Über die auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer strme gefhrt wird.*Ann. Phys. Chem* **72,**(1847), 497 - 508.

[4] R. Merris. Laplacian matrices of graphs, a survey, linear algebra and its applications. **197**,(1994), 143 - 176 .

[5] A. Modabish, and M. El Marraki, The number of spanning trees of certain families of planar maps, *Applied Mathematical Sciences,* **18** (2011), 883 - 898.

[6] Mohammad Hassan Shirdareh Haghighi, Recursive Relations for the Number of Spanning Trees, *Applied Mathematical Sciences,* **46** (2009), 2263 - 2269.

[7] Z. R. Bogdanowicz. Formulas for the number of spanning trees in a fan. Applied Mathematical Sciences, **2** (2008), 781 - 786.

[8] A. Modabish and M. El Marraki. Counting the Number of Spanning Trees in the Star Flower Planar Map. Applied Mathematical Sciences, **6** (2012), 2411 - 2418.

[9] D. Lotfi and M. El Marraki. Recursive relation for counting complexity of butterfly map. Journal of Theoretical & Applied Information Technology, Issue 1, p43, **29** (2011), 1817-3195.

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