Applied Mathematical Sciences, Vol. 9, 2015, no. 65, 3193 - 3203 HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.54307

Dynamics of Delayed Prey-Predator Model with Parental Care for Predators

M. Senthilkumaran

Department of Mathematics
Thiagarajar College
Madurai-625009, India

C. Gunasundari

Department of Mathematics
Fatima College
Madurai-625018, India

Abstract

In this paper, a stage structured prey-predator model (stage structure on predators) with two discrete time delays has been discussed. It is assumed that immature predators are raised by their parents in the sense that they cannot catch the prey and their foods are provided by parents. We suppose that the growth of the prey is of exponential type. The two discrete time delays occur due to maturation delay and gestation delay. Linear stability analysis for both non delay as well as with delays reveals that certain thresholds have to be maintained for coexistence. Numerical simulation shows that the system exhibits Hopf bifurcation, resulting in a stable limit cycle.

Keywords: Stage structure, Maturation delay, Gestation delay, Hopf bifurcation

1. Introduction

Prey- Predator systems are very important in the models of multispecies population dynamics and have been studied by many authors [1], [2], [3], [5],

[7]. In the natural world, there are many species whose individuals have a life history that take them through two stages, immature and mature, where immature predators are raised by their parents and the rate they attack at prey and the reproductive rate can be ignored. Stage structured models have recieved much attention in recent years. Recently Wang W, Takeuchi Y, Saito Y, Nakaoka S [6] studied the following predator prey system with parental care for predators.

$$\begin{aligned} \dot{x}(t) &= xg(x) - \beta x y_2 \\ \dot{y}_1(t) &= k_1 \beta x y_2 \frac{y_2}{wy_1 + y_2} \quad d_1 y_1 - k_2 \beta x y_2 \frac{wy_1}{wy_1 + y_2} \\ \dot{y}_2(t) &= k_2 \beta x y_2 \frac{wy_1}{wy_1 + y_2} \quad d_2 y_2 \end{aligned} \tag{1}$$

where x represents the prey, y_1 and y_2 represents immature and mature predators respectively, k_1 is a conversion coefficient, d_1 and d_2 are death rate of immature and mature predators respectively, and $g(x)$ is the per capita birth rate of the prey. In general, delay differential equations exhibit much more complicated systems than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate.

Motivated by the work of Sandip banerjee, B. Mukopadhyay and R.Bhattacharya [4], in the present paper we incorporate two discrete time delays in system (1) to make the model more realistic as follows.

$$\begin{aligned} \dot{x}(t) &= xg(x) - \beta x y_2(t - \tau_1) \\ \dot{y}_1(t) &= k_1 \beta x y_2 \frac{y_2}{wy_1 + y_2} \quad d_1 y_1 - k_2 \beta x y_2 \frac{wy_1}{wy_1 + y_2} \\ \dot{y}_2(t) &= k_2 \beta x(t - \tau_2) y_2 \frac{wy_1}{wy_1 + y_2} \quad d_2 y_2 \end{aligned} \tag{2}$$

where $\tau_1 \geq 0$ is called the gestation delay and $\tau_2 \geq 0$ is the delay in the

$$x(\theta) = \varphi(\theta) \geq 0, y_1(\theta) = \psi_1(\theta) \geq 0, y_2(\theta) = \psi_2(\theta) \geq 0$$

predator maturation. The initial conditions for the system takes the form

$$\varphi(0) > 0, \psi_1(0) > 0, \psi_2(0) > 0 \tag{3}$$

where $\tau = \max\{\tau_1, \tau_2\}$, $(\varphi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], R^3_{+0})$, the banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_{+0} , where $R_{3+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$ as the interior of R_{3+0}.

The remainder of this paper is organised as follows. In section 2 we discussed about the positivity and boundedness of the system. Section 3 shows the inherent exponential growth of prey. We try to interpret our results by numerical simulation in section 4. A brief discussion is given in section 5 to conclude this work.

2. Positivity and boundedness

Theorem 2.1

Every solution of system (2) with initial conditions (3) is bounded for all $t > 0$ and all of these solutions are ultimately bounded.

Definition 2.1:

A system is said to be permanent if there exists a compact region $\Omega_0 \in$ int R^3_+ such that every solution of system with initial conditions will eventually enter and remain in region Ω_0.

Theorem 2.2:

System (2) is permanent provided that $g(x) - \beta M_2 > 0$.

3. Inherent exponential growth of prey

In this section, we suppose that the growth of the prey is of exponential type in the absence of predation.We show that the inclusion of the stage structure for predators can stabilize or destabilize the dynamics of Prey- Predator interactions, depending on the combination of biological parameters. Note that (2) is reduced to

$$\begin{aligned} \dot{x}(t) &= rx - \beta x y_2(t-\tau_1) \\ \dot{y}_1(t) &= k_1\beta x y_2 \frac{y_2}{wy_1+y_2} - d_1 y_1 - k_2\beta x y_2 \frac{wy_1}{wy_1+y_2} \\ \dot{y}_2(t) &= k_2\beta x(t-\tau_2) y_2 \frac{wy_1}{wy_1+y_2} - d_2 y_2 \end{aligned} \tag{1}$$

3.1 Equilibria analysis

By introducing scaling variables $u = \frac{x\beta k_2}{d_2}$, $v_1 = \frac{y_1 w\beta}{d_2}$, $v_2 = \frac{y_2\beta}{d_2}$, $\theta = d_2 t$, and then still using old variables for simplicity in notations, we obtain

$$\begin{aligned} \dot{x}(t) &= bx - xy_2(t-\tau_1) \\ \dot{y}_1(t) &= k\frac{xy_2^2}{y_1+y_2} - dy_1 - w\frac{xy_2 y_1}{y_1+y_2} \end{aligned} \tag{2}$$

$y_1 + y_2$

$$\dot{y}_2(t) = \frac{x y_2 y_1}{y_1 + y_2}$$

where $b = \frac{d_2}{r}$, $k = \frac{wk_1}{k_2}$, $d = \frac{d_1}{d_2}$.
This system admits a unique positive equilibrium $E^* = (x^*, y^*_1, b)$ where $x^* = \frac{b+y^*_1}{y^*_1}$ and y^*_1 is the unique positive solution of the following equation.

$$dy^2_1 + bwy_1 - kb^2 = 0. \tag{3}$$

The Jacobian matrix of (2) at E^* is

$$\begin{vmatrix} -\lambda & 0 & -x^* e^{-\lambda\tau_1} \\ \frac{b(kb-wy^*_1)}{y^*_1+b} & -\frac{(kx^*b^2+dy^{*2}_1+2dy^*_1b+db^2+x^*b^2w)}{(y^*_1+b)^2} - \lambda & \frac{x^*(2kby^*_1+kb^2-wy^{*2}_1)}{(y^*_1+b)^2} \\ \frac{y^*_1 b e^{-\lambda\tau_2}}{(y^*_1+b)} & \frac{x^*b^2}{(y^*_1+b)^2} & -\frac{x^*y^*_1 b}{(y^*_1+b)^2} - \lambda \end{vmatrix} = 0, \tag{4}$$

The characteristic equation of the Jacobian matrix is

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D_1\lambda + E_1)e^{-\lambda\tau_1} + (D_2\lambda + E_2)e^{-\lambda\tau_2} + (F\lambda + G)e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{5}$$

where $A = \frac{(kx^*b^2+dy^{*2}_1+2dy^*_1b+db^2+x^*b^2w+x^*y^*_1b)}{(y^*_1+b)^2}$,

$B = \frac{(kx^*b^2+dy^{*2}_1+2dy^*_1b+db^2+x^*b^2w)x^*y^*_1b-2kb^3y^*_1x^{*2}-kb^4x^{*2}+wy^{*2}_1x^{*2}b^2}{(y^*_1+b)^4}$,

$C = 0$,

$D_1 = 0, E_1 = \frac{x^{*2}b^3(kb-wy^*_1)}{(y^*_1+b)^3}$

$D_2 = 0$,
$E_2 = 0$,
$F = \frac{x^*by^*_1}{y^*_1+b}$
$G = \frac{x^*y^*_1b(kx^*b^2+dy^{*2}_1+2dy^*_1b+db^2+x^*b^2w)}{(y^*_1+b)^3}$

Case 1: $\tau_1 = 0, \tau_2 = 0$

In this case the characteristic equation (5) reduces to

$$\lambda^3 + A\lambda^2 + (B + D_1 + D_2 + F)\lambda + (C + E_1 + E_2 + G) = 0 \tag{6}$$

Theorem 3.1

Assume that (H_1) $(B + D_1 + D_2 + F) > 0$, $(C + E_1 + E_2 + G) > 0$. Then the system (2) without delay will be locally asymptotically stable around $E^* = (x^*, y^*_1, y^*_2)$.

Case 2: $\tau_1 = 0$, $\tau_2 > 0$

In this case the characteristic equation (5) reduces to

$$\lambda^3 + A\lambda^2 + (B + D_1)\lambda + (C + E_1) + ((D_2 + F)\lambda + E_2 + G)e^{-\lambda\tau_2} = 0 \tag{7}$$

Lemma 3.1

For $\tau_1 = 0$, assume that H_1 is satisfied. Then the following conclusion holds.

1. If $A^2 - 2(B + D_1) > 0$, $(B + D_1)^2 - (D_2 + F)^2 - 2A(C + E_1) > 0$, $(C + E_1)^2 - (E_2 + G)^2 > 0$ holds, then equilibrium (x^*, y^*_1, y^*_2) is asymptotically stable for all $\tau_2 > 0$.
2. If $(C + E_1)^2 - (E_2 + G)^2 < 0$ holds, then equilibrium (x^*, y^*_1, y^*_2) is asymptotically stable for $\tau_2 < \tau_{20}$, and unstable for $\tau_2 > \tau_{20}$. Furthermore, system undergoes a Hopf bifurcation at (x^*, y^*_1, y^*_2) when $\tau_2 = \tau_{20}$.
3. If $2(B + D_1) - A^2 > 0$, $(D_2 + F)^2 - (B + D_1) + 2A(C + E_1) > 0$, $(C + E_1)^2 - (E_2 + G)^2 > 0$ and $[(D_2 + F)^2 - (B + D_1)^2 - 2A(C + E_1)]^2 > 4[(C + E_1)^2 - (E_2 + G)^2]$ holds, then there exists a positive integer m such that the equilibrium is stable when $\tau_2 \in [0, \tau_{20}^{+}) \cup (\tau_{20}^{-}, \tau_{21}^{+}) \cup \dots \cup (\tau_{2m-1}^{-}, \tau_{2m}^{+})$ and unstable when $\tau_2 \in [\tau_{20}^{+}, \tau_{20}^{-}) \cup (\tau_{21}^{+}, \tau_{21}^{-}) \cup \dots \cup (\tau_{2m}^{+}, \tau_{2m}^{-}) \cup (\tau_{2m}^{+}, \infty)$

Case 3: $\tau_1 > 0$, $\tau_2 = 0$

In this case the characteristic equation (5) reduces to

$$\lambda^3 + A\lambda^2 + (B + D_2)\lambda + (C + E_2) + ((D_1 + F)\lambda + (E_1 + G))e^{-\lambda\tau_1} = 0 \quad (8)$$

Let $i\omega(\omega > 0)$ be a root of the equation (8) , then

$$-i\omega^3 - A\omega^2 + (B + D_2)i\omega + (C + E_2) + ((D_1 + F)i\omega + (E_1 + G))e^{-i\omega\tau_1} = 0 \quad (9)$$

Equating real and imaginary parts,

$$(E_1 + G)\cos(\omega\tau_1) + (D_1 + F)\,\omega\,\sin(\omega\tau_1) = A\omega^2 - (C + E_2) \quad (10)$$

$$(D_1 + F)\,\omega\,\cos(\omega\tau_1) - (E_1 + G)\sin(\omega\tau_1) = \omega^3 - (B + D_2)\omega \quad (11)$$

which implies

$$\cos(\omega\tau_1) = \frac{[(D_1 + F)\omega^4 + (A(E_1 + G) - (D_1 + F)(B + D_2))\omega^2 - ((E_1 + G)(C + E_2))]}{(E_1 + G)^2 + \omega^2(D_1 + F)^2} \quad (12)$$

$$\sin(\omega\tau_1) = \frac{[[A(D_2 + F) - (E_2 + G)]\omega^3 + ((E_2 + G)(B + D_1) - (D_2 + F)(C + E_1))\omega]}{(D_2 + F)^2\omega^2 + (E_2 + G)^2} \quad (13)$$

Squaring and adding we get

$$\omega^6 + \omega^4(A^2 - 2(B + D_2)) + [(B + D_2)^2 + 2A(C + E_2) - (D_1 + F)^2]\omega^2 + (C + E_2)^2 - (E_1 + G)^2 = 0 \quad (14)$$

Let

$$\psi(W) \equiv W^3 + W^2(A^2 - 2(B + D_2)) + [(B + D_2)^2 + 2A(C + E_2) - (D_1 + F)^2]W + (C + E_2)^2 - (E_1 + G)^2 = 0 \quad (15)$$

where $W = \omega^2$.

The function 0 has positive roots iff

$$(C + E_2)^2 - (E_1 + G)^2 < 0,$$

Without loss of generality, let W_p be the positive roots of $0 = 0$ and let $\omega_p = \sqrt{W_p}$. We note that the unique solution of $\theta = [0, 2\pi]$ of (12) and (13) is

$$= \cos^{-1}\left(\frac{(D_1 + F)\omega^4 + (A(E_1 + G) - (D_1 + F)(B + D_2))\omega^2 - ((E_1 + G)(C + E_2))}{(E_1+G)^2+\omega^2(D_1 +F)^2}\right) \quad (16)$$

if $sin(\theta) > 0$, that is, if $(A(D_2 + F) - (E_2 +G))\omega^2 + (E_2 +G)(B + D_1) - (D_2 + F)(C + E_1) > 0$ and

$$\theta = 2\pi - \cos^{-1}\left(\frac{(D_1 + F)\omega^4 + (A(E_1 + G) - (D_1 + F)(B + D_2))\omega^2 - ((E_1 + G)(C + E_2))}{(E_1+G)^2+\omega^2(D_1 +F)^2}\right) \quad (17)$$

if $(A(D_2 + F) - (E_2 + G))\omega^2 + (E_2 + G)(B + D_1) - (D_2 + F)(C + E_1) \leq 0$. We now define two sequences,

$$\tau_{1,p}^{1,i} = \frac{1}{\omega_p}\left[\cos^{-1}\left(\frac{(D_1 + F)\omega^4 + (A(E_1 + G) - (D_1 + F)(B + D_2))\omega^2 - ((E_1 + G)(C + E_2))}{(E_1+G)^2+\omega^2(D_1+F)^2}\right) + 2i\pi\right]$$

$$\tau_{1,p}^{2,i} = \frac{1}{\omega_p}\left[2\pi - \cos^{-1}\left(\frac{(D_1 + F)\omega^4 + (A(E_1 + G) - (D_1 + F)(B + D_2))\omega^2 - ((E_1 + G)(C + E_2))}{(E_1+G)^2+\omega^2(D_1 +F)^2}\right) + 2i\pi\right]$$

Theorem 3.2

Let $T^*_{1,p} = T^{1,i}_{1,p}$ or $T^*_{1,p} = T^{2,i}_{1,p}$, that is $T^*_{1,p}$ represents an element either of the sequence $T^{1}_{1,p}$ or $T^{2}_{1,p}$ associated with ω_p. Then the equation $\lambda^3 + A\lambda^2 + (B + D_2)\lambda + (C + E_2) + ((D_1 + F)\lambda + (E_1 + G))e^{-\lambda\tau_1} = 0$ has a pair of simple conjugate roots $\pm i\omega_p$ for $T_2 = T^*_{1,p}$ which satisfies

$$\frac{d\,Re\lambda}{dT}\Big|_{\tau=\tau_{1,p}} = sign\,0(\omega^2 p) \quad (18)$$

Denoting $T_i = min_{i\in N}$ $T_{1,p}$, it is concluded that the steady state (x_*, y^*_1, y^*_2) is locally asymptotically stable if $T_1 = T^*_1$ iff $'0(\omega^2 p) > 0$.

Case 4: $T_1 > 0$, $T_2 > 0$

We now state a result regarding the sign of the real parts of the roots (5) in order to study the local stability of the positive steady state (x_*, y^*_1, y^*_2) of system (2).

Proposition (P_1):

If all the roots of the equation (5) have negative real parts for some $T_1 > 0$, then there exists a $T^*_2(T_1) > 0$ such that all the roots of equation (5) (i.e with $T_2 > 0$) have negative real parts when $T_2 < T^*_2(T_1)$.

Considering the above proposition we can now state the following theorem.

Theorem 3.3

If we assume that the hypothesis P_1 hold, then for any $T_1 \in [0, T^*_1)$, there exists a $T^*_2(T_1) > 0$ such that the positive steady state (x^*, y^*_1, y^*_2) of the sytem is locally asymptotically stable when $T_1 \in [0, T^*_1)$.

Proof:

Using the above theorem, we can say that all the roots of (5) have negative real parts when $T_1 \in [0, T^*_1)$ and by proposition we can conclude that there exists a $T^*_2(T_1) > 0$ such that all the roots of equation (5) have negative real parts when $T_2 < T^*_2(T_1)$. Hence the steady state (x^*, y^*_1, y^*_2) of system (2) is locally asymptotically stable when $T_1 \in [0, T^*_1)$.

4. Numerical Simulation

In this section, we present some numerical results of the system (2) to verify the analytical predictions obtained in previous section. Let us consider the system (2) with hypothetical set parameter values d = 0.5, w = 0.5, k = 0.1, b = 2. Then

$$\dot{x}(t) = 2x - xy_2(t - T_1)$$
$$\dot{y}_1(t) = 0.1\,\frac{xy_2^2}{y_1 + y_2} - 0.5y_1 - 0.5\,\frac{xy_2y_1}{y_1 + y_2} \quad (1)$$
$$\dot{y}_2(t) = \frac{x(t - T_2)y_2y_1}{y_1 + y_2}\; y_2$$

which has a positive equilibrium $E^*(x, y_1, y_2) = (6.8541, 0.3416, 2)$. When $T_1 = 0$, $T_2 = 0$, the equilibrium E^* is asymptotically stable if d < 1 and is unstable if d > 1. Fig 1 shows that the positive solutions of (1) approach E^* is an oscillatory form if E^* is stable. Hence less mortality rate of juvenile predators relative to that of adult predators has a stabilizing effect and a larger one destabilizes the equilibrium and produces cycle. Fig 2 shows the time evolution of three populations when time of maturity is very small ($T_2 = 0.015$) and no gestation delay ($T_1 = 0$). It shows that the three populations are asymptotically stable.

Fig1: r1 = 0andr2=0

$(x1\ (0), y1\ (0), y2\ (0))^T = (1, 2, 2.2)^T$

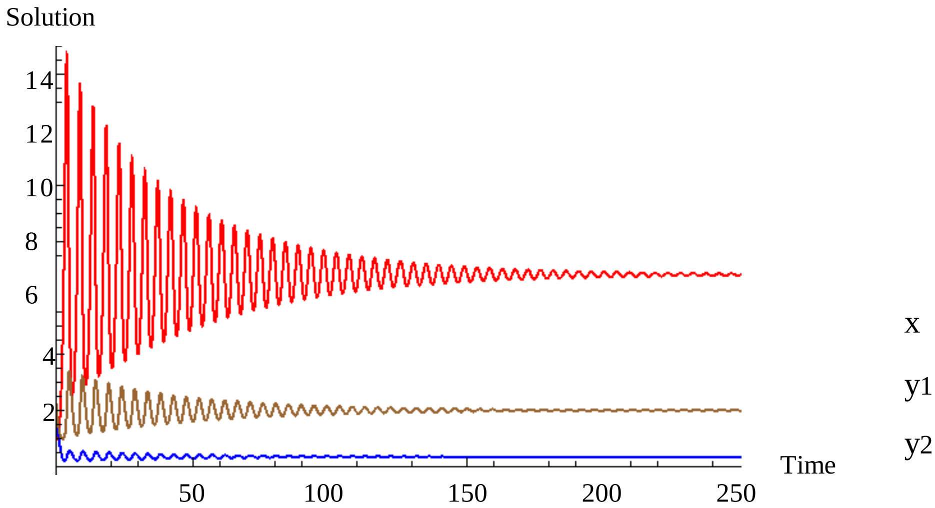

Fig 2: t1 = 0 and t2 = 0.015

Hx1HqL,y1HqL,y2HqLL^T =H1, 1.5, 2L^T

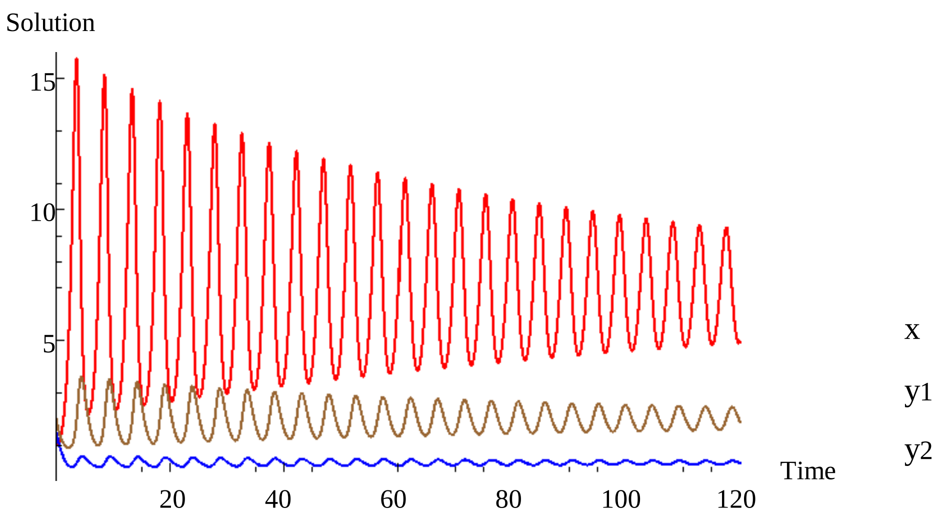

Fig 3: r1 = 0.012 and r2 = 0

$(x1(0), y1(0), y2(0))^T = (1, 1.7, 2.1)^T$

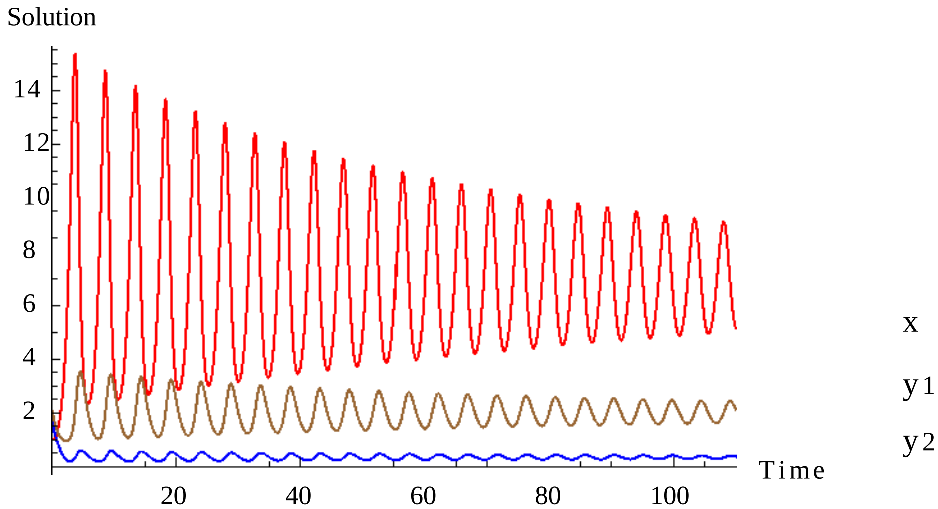

Fig 4: t1 = 0.013 and t2 = 0.01

Hx1 HqL,y1 HqL,y2HqLL^T =H1, 1.8, 2.2L^T

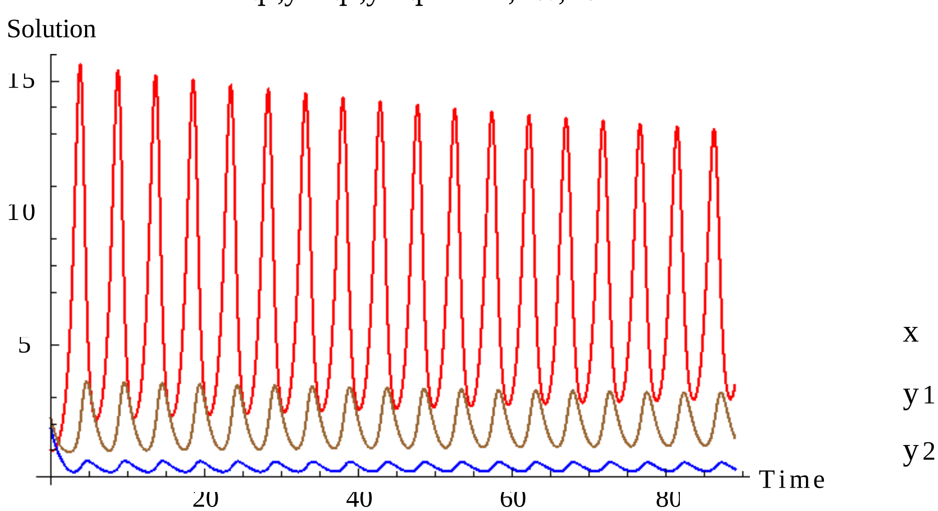

Fig 3 gives the time evolution of the three populations when gestation delay is taken into account where maturation delay is zero. The positive equilibrium E_* is asymptotically stable for ($r_1 = 0.012$) and ($r_1 = 0$). Fig 4 shows that the steady state is asymptotically stable, though damped oscillations can be observed. The time delays are ($r_1 = 0.013$), ,($r_2 = 0.01$).

5. Discussion

In this paper, we have considered the resource availability for immature predators. We have studied the case where the food of immature predators is given by their parents(adult predators). It is assumed that the mature predators consume the resource for their growth. We have studied the effect of two time delays on the dynamical behavior of a prey predator system with stage structure for predators. For non delay case, if the prey grows in the form of exponential type and the transition rate is the linear function of the nutrient availability to one immature predator in unit time, we have found that the ratio $d = d_1/d_2$ determines the stability. Then we have considered the delay case . When there is no gestation delay , the maturation time plays an important role in the dynamics of the system. For small maturity time of the prey species, all the population perish. In the presence of gestation and maturation delay, due to the gestation of prey biomass by predators, an oscillatory behavior of both prey and predator populations is noted. This indicates that seasonal effects on population models often lead to synchronous solutions.

References

[1] H. J. Hu, L. H. Huang, Stability and Hopf bifurcation in a delayed predator-prey system with stage structure for prey, *Nonlinear Anal. Real World Appl.* **11(4)** (2010) 2757-2769. http://dx.doi.org/10.1016/j.nonrwa.2009.10.001

[2] S. Liu, L. Chen, Z. Liu, Extinction and permanence in nonautonomous competitive system with stage structure, *J. Math. Anal. Appl.* **274** (2002) 667-684. http://dx.doi.org/10.1016/s0022-247x(02)00329-3

[3] J. G. Magnusson, Destability effect of cannibalism on a structured predator-prey system, *Math. Biol.* **101** (1990) 139- 153.

[4] Sandip Banerjee, B. Mukhopadhyay and R. Bhattacharyya, Effect of maturation and gestation delays in a stage structure predator prey model, *J. Appl. Math. Informatics* **Vol. 28** (2010), No. 5-6, pp. 1379-1393.

[5] W. Wang, G. Mulone, F. Salemi, V. Salone, Permanence and stability of a stage-structured predator-prey model, *J. Math. Anal. Appl.* ***262 (2)*** (2001) 499-528. http://dx.doi.org/10.1006/jmaa.2001.7543

[6] W. Wang, Y. Takeuchi, Y. Saito, S. Nakaoka, Prey-Predator system with parental care for predators, *J Theor Biol.* 2006 Aug 7; ***241(3): 451-8.*** Epub 2006 jan 18. http://dx.doi.org/10.1016/j.jtbi.2005.12.008

[7] F. Wei, K. Wang, Permanence of variable coefficients predator-prey system with stage *structure,Appl.Math.Comput.* ***180*** (2006) 594-598. http://dx.doi.org/10.1016/j.amc.2005.12.062

Received: December *15, 2014;* Published: April *20, 2015*